Non-Markovian disentanglement dynamics of a two-qubit system

Xiufeng Cao* and Hang Zheng

Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

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We investigate the disentanglement dynamics of a two-qubit system in the non-Markovian approach. It is shown that only for weak coupling between the system and environment does an exponential decay of entanglement appear, for certain classes of two-qubit entangled states. When the coupling between qubit and the environment becomes stronger, entanglement sudden death always appears even if the dissipation environment is at zero temperature.

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I. INTRODUCTION

A multipartite quantum system, in addition to local quantum coherence that exists within each of the subsystems, may have nonlocal or distributed quantum coherence that exists among several distinct subsystems. This property is entanglement, which is superposition of the internal states of the subsystems and cannot be separated into product states of the individual subsystem. It is recognized as an entirely quantum-mechanical effect and has played a crucial role in practical applications ranging from quantum information [1,2], cryptography [3], and quantum computation [4,5] to atomic and molecular spectroscopy [6,7].

Recently, many groups were able to prepare entangled states in a variety of physical systems or experimental setups, demonstrating an impressive ability to manipulate and detect them efficiently [8–15]. In particular, Almeida *et al.* [15] showed that, using an all-optical experimental setup, even when the environment-induced decay of each subsystem is asymptotic, "entanglement sudden death" (ESD [18]) may appear, that is, entanglement terminates completely after a finite interval without a smoothly diminishing long-time tail.

As we know, a major obstacle for keeping entanglement of subsystems remains with the absence of capacity to perfectly screen of the system from the environment. After some time, the unavoidable residual interaction with the environment induces mixing of the system states, and thus the emergence of classical correlations at the expense of quantum entanglement. Hence, it is desirable to understand how the entanglement decays and what the associated time scale is. Enlightened by the experimental discovery of ESD, a large number of theoretical papers have investigated the disentanglement dynamics [16–19]. Ficek and Tanaś [16] give an overview of the mathematical formalism necessary for describing the interaction of atoms with the electromagnetic field. Mintert *et al.* [17] start with a short recollection of environment models adapted for decoherence process in a typical quantum optical context under the assumption of Markovian dynamics in the Lindblad formalism. Yu and Eberly [18,19] showed that the dynamics of the quantum entanglement between two qubits interacting independently with either quantum noise or classical noise displays a completely different behavior from the dynamics of the local coherence. Instead of the exponential decay in time of the local coherence, quantum entanglement may disappear within a finite time in the dynamical evolution. Among these methods, it is surprising that few if any fundamental treatments of decoherence exist that include the dynamics of disentanglement with better than the Markovian approximation or phenomenological tradition. Although, the use of the Markovian approximation is justified in a large variety of quantum optical experiments where entanglement has been produced, one should notice that non-Markovian effects are important in describing some condensed-matter systems [17,20], such as the quantum dot qubit(s) system. Therefore, it is desirable to study a non-Markovian effects of the decoherence and disentanglement in any viable realization of qubits.

In this paper we examine the disentanglement dynamics of two qubits due to spontaneous emission, without rotating-wave approximation in the interaction with the environment and without Markovian approximation in the time evolution. It is found that disentanglement always takes place within a finite time, called "entanglement sudden death," when the coupling between qubit and environment is not weak. While the coupling with dissipation environments becomes weak, the disentanglement changes from exponential decay to entanglement sudden death with the increasing of the portion of the double excitation component in the initial entangled state. We describe the lifetime of ESD through the measurable parameters: coupling constant with the environment α, energy splitting Δ, and cutoff frequency ω_c. Since the entanglement plays an important role in quantum information processing, the time scale of the disentanglement dynamics in the open environment should be one of the most relevant problems in both theoretical and experimental studies.

The paper is organized as follows. In Sec. II we introduce the Hamiltonian without rotating-wave approximation in the qubit-environment interaction, and solve it in terms of a non-Markovian approach based on unitary transformation. The dependence of the concurrence on the different initial conditions and the coupling strength to the dissipation environment are discussed in Sec. III. Finally, the conclusion is given in Sec. IV.

*cxf@sjtu.edu.cn

II. THE MODEL AND THEORY

This paper is concerned with a pair of two-level systems, since it is generally believed that entanglement of only two microscopic quantum systems (qubits, atoms) is essential to implement quantum protocols such as quantum computation. We consider two subsystems A, B and assume that each subsystem interacts independently with its environments, a well justified assumption wherever the particles composing your system are sufficiently separated from each other, and therefore, no collective environment effects must be taken into account. Such a model in nonrotating wave form may be formulated to the following Hamiltonian (set $\hbar=1$):

$$H = H_{\mathrm{qu}} + H_{\mathrm{env}} + H_{\mathrm{int}}, \tag{1}$$

with

$$H_{\mathrm{qu}} = -\frac{1}{2}\Delta_A\sigma_z^A - \frac{1}{2}\Delta_B\sigma_z^B, \tag{2}$$

$$H_{\mathrm{env}} = \sum_k \omega_k a_k^\dagger a_k + \sum_k \nu_k b_k^\dagger b_k, \tag{3}$$

$$H_{\mathrm{int}} = \frac{1}{2}\sum_k g_k(a_k^\dagger + a_k)\sigma_x^A + \frac{1}{2}\sum_k f_k(b_k^\dagger + b_k)\sigma_x^B, \tag{4}$$

where the Hamiltonian of the two qubits is H_{qu}, that of the two independent environments is H_{env}, and that of the interaction is H_{int}. Here σ_i $(i=x,y,z)$ denotes the usual Pauli spin matrices, Δ_A (Δ_B) describes the energy splitting in the A (B) qubit. $a_k^\dagger$ $(b_k^\dagger)$, a_k (b_k), and ω_k (ν_k) are the creation, annihilation operators, and energy with wave vector k in the A (B) qubit environment. g_k and f_k are the qubit-environment coupling strengths. Yu and Eberly [18,19] have employed a similar model, in which, however, the rotating-wave approximation is valid. Two environments are completely defined by the spectral density:

$$J(\omega) = \sum_k g_k^2\delta(\omega - \omega_k). \tag{5}$$

We consider the Ohmic bath $J(\omega)=2\alpha\omega\theta(\omega_c-\omega)$ in this work, where α is the dimensionless coupling constant and $\theta(x)$ is the usual step function.

In order to simplify the non rotating-wave term, we apply a canonical transformation $H'=\exp(S)H\exp(-S)$ with the generator [21]

$$S = \sum_k \frac{g_k}{2\omega_k}\xi_k^A(a_k^\dagger - a_k)\sigma_x^A + \sum_k \frac{f_k}{2\nu_k}\xi_k^B(b_k^\dagger - b_k)\sigma_x^B. \tag{6}$$

Then decompose the transformed Hamiltonian H' into three parts:

$$H' = H_0' + H_1' + H_2'. \tag{7}$$

The three parts include the analogous form of the A and B qubits

$$H_0' = H_{0A}' + H_{0B}' \tag{8}$$

with

$$H_{0A}' = -\frac{1}{2}\eta^A\Delta_A\sigma_z^A + \sum_k \omega_k a_k^\dagger a_k - \sum_k \frac{g_k^2}{4\omega_k}\xi_k^A(2-\xi_k^A), \tag{9}$$

$$H_{0B}' = -\frac{1}{2}\eta^B\Delta_B\sigma_z^B + \sum_k \nu_k b_k^\dagger b_k - \sum_k \frac{f_k^2}{4\nu_k}\xi_k^B(2-\xi_k^B). \tag{10}$$

In the same way, $H_1'=H_{1A}'+H_{1B}'$ and $H_2'=H_{2A}'+H_{2B}'$, where

$$H_{1A}' = \sum_k \eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}(a_k^\dagger\sigma_-^A + a_k^-\sigma_+^A), \tag{11}$$

$$\begin{aligned} H_{2A}' = &-\frac{1}{2}\Delta_A\sigma_z^A\left\{\cosh\left[\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right] - \eta^A\right\} \\ &- i\frac{\Delta_A}{2}\sigma_y^A\left\{\sinh\left[\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right]\right. \\ &\left. - \eta^A\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right\}, \end{aligned} \tag{12}$$

with

$$\eta^A = \exp\left[-\sum_k \frac{g_k^2}{2\omega_k^2}(\xi_k^A)^2\right], \quad \xi_k^A = \frac{\omega_k}{\omega_k + \eta^A\Delta_A}, \tag{13}$$

$$\eta^B = \exp\left[-\sum_k \frac{f_k^2}{2\nu_k^2}(\xi_k^B)^2\right], \quad \xi_k^B = \frac{\nu_k}{\nu_k + \eta^B\Delta_B}, \tag{14}$$

correspondingly. Here $\sigma_\pm^A=(\sigma_x^A \mp i\sigma_y^A)/2$, H_0' is the Hamiltonian of the noninteracting qubits and environments, H_1' and H_2' are the interaction Hamiltonian in increasing order of the qubit-environment coupling strength g_k and f_k. Comparing H_1 to H_1', the term in H_1 is replaced by the similar rotating-wave approximation term in H_1', while the qubit-environment coupling strength g_k in H_1 is replaced by $g_k\eta^A\Delta_A/(\omega_k+\eta^A\Delta_A)$ in H_1'. As we can see, $g_k\eta^A\Delta_A/(\omega_k+\eta^A\Delta_A)<g_k$, that is to say, the counter-rotating terms decrease the coupling strength with the environment.

Approximately, to order g_k^2 and f_k^2, we write the total Hamiltonian as $H'=H_0'+H_1'$. In the interaction picture

$$\begin{aligned} V_I'(t) = &\sum_k \eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}a_k^\dagger\sigma_-^A\exp[i(\omega_k - \eta^A\Delta_A)t] \\ &+ \sum_k \eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}a_k\sigma_+^A\exp[-i(\omega_k - \eta^A\Delta_A)t] \\ &+ \sum_k \eta^B\Delta_B\frac{f_k\xi_k^B}{\nu_k}b_k^\dagger\sigma_-^B\exp[i(\nu_k - \eta^B\Delta_B)t] \\ &+ \sum_k \eta^B\Delta_B\frac{f_k\xi_k^B}{\nu_k}b_k\sigma_+^B\exp[-i(\nu_k - \eta^B\Delta_B)t]. \end{aligned} \tag{15}$$

Next, we consider in general a system denoted by S interacting with a reservoir or environment denoted by R. The combined density operator is denoted by ρ_{SR}. The reduced density operator for the system ρ_S is obtained by taking a trace over the environment coordinates, i.e., $\rho_S = \mathrm{Tr}_R(\rho_{SR})$. The equation of motion for ρ_{SR} is given by

$$\frac{d}{dt}\rho^I_{SR}(t) = -i[V_I(t), \rho^I_{SR}(t)]. \tag{16}$$

After S transformation,

$$\frac{d}{dt}\rho^{I'}_{SR}(t) = -i[V'_I(t), \rho^{I'}_{SR}(t)]. \tag{17}$$

This equation can be formally integrated, and we obtain

$$\rho^{I'}_{SR}(t) = \rho^{I'}_{SR}(t_i) - i\int_{t_i}^{t}[V'_I(t'), \rho^{I'}_{SR}(t')]dt'. \tag{18}$$

Here t_i is an initial time when the interaction starts, supposing $t_i=0$. Substituting $\rho^{I'}_{SR}(t)$ into Eq. (17), we find the equation of motion

$$\frac{d}{dt}\rho^{I'}_{SR}(t) = -i[V'_I(t), \rho'_{SR}(0)] - \int_0^t \{V'_I(t), [V'_I(t'), \rho^{I'}_{SR}(t')]\}dt'. \tag{19}$$

We now employ the Born approximation [16,17,22] in which the interaction between the qubit and the environment is suppose to weak, and there is no back reaction effect of the qubit on the environment. In this approximation, the state of the environment does not change in time, and we can write the density operator $\rho'_{SR}(t)$ as $\rho'_{SR}(t) = \rho'_S(t)\rho'_R(0)$. Under this approximation, Eq. (19) simplifies to

$$\frac{d}{dt}\rho^{I'}_{SR}(t)\rho'_R(0) = -i[V'_I(t), \rho'_S(0)\rho'_R(0)] - \int_0^t \{V'_I(t), [V'_I(t'), \rho^{I'}_{SR}(t')\rho_R(0)]\}dt'. \tag{20}$$

Substituting $V'_I(t)$ into Eq. (20) and assuming the two environments' modes in thermalization, the Tr_R are given by

$$\mathrm{Tr}_R[a_k^\dagger a_k \rho_R] = \mathrm{Tr}_R[a_k \rho_R a_k^\dagger] = \mathrm{Tr}_R[b_k^\dagger b_k \rho_R] = \mathrm{Tr}_R[b_k \rho_R b_k^\dagger] = n_k, \tag{21}$$

$$\begin{aligned}\mathrm{Tr}_R[a_k a_k^\dagger \rho_R] &= \mathrm{Tr}_R[a_k^\dagger \rho_R a_k] \\ &= \mathrm{Tr}_R[b_k b_k^\dagger \rho_R] \\ &= \mathrm{Tr}_R[b_k^\dagger \rho_R b_k] \\ &= n_k + 1.\end{aligned} \tag{22}$$

Then,

$$\begin{aligned}\frac{d}{dt}\rho^{I'}_S(t) = &-\int_0^t \sum_k \left(\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}\right)^2 n_k^A[\sigma_-^A\sigma_+^A\rho_S^{I'}(t') - \sigma_+^A\rho_S^{I'}(t')\sigma_-^A]\exp[i(\omega_k - \eta^A\Delta_A)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}\right)^2 (n_k^A+1)[\rho_S^{I'}(t')\sigma_+^A\sigma_-^A - \sigma_-^A\rho_S^{I'}(t')\sigma_+^A]\exp[i(\omega_k - \eta^A\Delta_A)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}\right)^2 n_k^A[\rho_S^{I'}(t')\sigma_-^A\sigma_+^A - \sigma_+^A\rho_S^{I'}(t')\sigma_-^A]\exp[-i(\omega_k - \eta^A\Delta_A)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}\right)^2 (n_k^A+1)[\sigma_+^A\sigma_-^A\rho_S^{I'}(t') - \sigma_-^A\rho_S^{I'}(t')\sigma_+^A]\exp[-i(\omega_k - \eta^A\Delta_A)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k}\right)^2 n_k^B[\sigma_-^B\sigma_+^B\rho_S^{I'}(t') - \sigma_+^B\rho_S^{I'}(t')\sigma_-^B]\exp[i(\nu_k - \eta^B\Delta_B)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k}\right)^2 (n_k^B+1)[\rho_S^{\prime I}(t')\sigma_+^B\sigma_-^B - \sigma_-^B\rho_S^{\prime I}(t')\sigma_+^B]\exp[i(\nu_k - \eta^B\Delta_B)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k}\right)^2 n_k^B[\rho_S^{I'}(t')\sigma_-^B\sigma_+^B - \sigma_+^B\rho_S^{I'}(t')\sigma_-^B]\exp[-i(\nu_k - \eta^B\Delta_B)(t-t')]dt' \\ &-\int_0^t \sum_k \left(\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k}\right)^2 (n_k^B+1)[\sigma_+^B\sigma_-^B\rho_S^{I'}(t') - \sigma_-^B\rho_S^{I'}(t')\sigma_+^B]\exp[-i(\nu_k - \eta^B\Delta_B)(t-t')]dt'.\end{aligned} \tag{23}$$

In this equation, the n_k and n_k+1 terms on the right-hand side describe, respectively, decay and excitation processes, with rates which depend on the temperature, here parameterized by n_k, the average thermal excitation of the environment. In this work, we study the limit of zero temperature $n_k=0$, that is to say only the spontaneous decay term survives leading to purely dissipative process.

The matrix equation is solved in the space spanned by the two-qubit product states basis $|1\rangle=|\uparrow\uparrow\rangle, |2\rangle=|\uparrow\downarrow\rangle, |3\rangle=|\downarrow\uparrow\rangle, |4\rangle=|\downarrow\downarrow\rangle$, where $\sigma_z|\uparrow\rangle=(+1)|\uparrow\rangle$, $\sigma_z|\downarrow\rangle=(-1)|\downarrow\rangle$. After Laplace transformation and convolution operation, the master equation of the two qubits system can be obtained as follows [23]:

$$P\overline{\rho_S^{I\prime}(P)}-\rho_S'(0)=-\sum_k\frac{\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2}{P-i(\omega_k-\eta^A\Delta_A)}[\overline{\rho_S^{I\prime}(P)}\sigma_+^A\sigma_-^A-\sigma_-^A\overline{\rho_S^{I\prime}(P)}\sigma_+^A]-\sum_k\frac{\left(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k}\right)^2}{P-i(\nu_k-\eta^B\Delta_B)}[\overline{\rho_S^{I\prime}(P)}\sigma_+^B\sigma_-^B-\sigma_-^B\overline{\rho_S^{I\prime}(P)}\sigma_+^B]$$
$$-\sum_k\frac{\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2}{P+i(\omega_k-\eta^A\Delta_A)}[\sigma_+^A\sigma_-^A\overline{\rho_S^{I\prime}(P)}-\sigma_-^A\overline{\rho_S^{I\prime}(P)}\sigma_+^A]-\sum_k\frac{\left(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k}\right)^2}{P+i(\nu_k-\eta^B\Delta_B)}[\sigma_+^B\sigma_-^B\overline{\rho_S^{I\prime}(P)}-\sigma_-^B\overline{\rho_S^{I\prime}(P)}\sigma_+^B]. \quad (24)$$

Denote the summation of the environment degree of freedom $A_\pm=\Sigma_k\frac{[\eta^A\Delta_A(g_k\xi_k^A/\omega_k)]^2}{P\pm i(\omega_k-\eta^A\Delta_A)}$ and $B_\pm=\Sigma_k\frac{[\eta^B\Delta_B(g_k\xi_k^B/\nu_k)]^2}{P\pm i(\nu_k-\eta^B\Delta_B)}$. The decay rate and lamb shift are dependent on those processes, as seen from $A_\pm$ and $B_\pm$, instead of being constant for all processes in the Markovian approximation. We shall therefore focus on the precise time scales of every decay process.

According to the Kronecker product property and the technique of Lyapunov matrix equation in matrix theory, expand the matrix into vector along rows from two sides of the master equation

$$\begin{aligned}\{PI_{16\times16}&+[A_-I_{4\times4}\otimes(\sigma_+^A\sigma_-^A\otimes I_{2\times2})^T+B_-I_{4\times4}\otimes(I_{2\times2}\\&\otimes\sigma_+^B\sigma_-^B)^T]-[A_-(\sigma_-^A\otimes I_{2\times2})\otimes(\sigma_+^A\otimes I_{2\times2})^T+B_-(I_{2\times2}\\&\otimes\sigma_-^B)\otimes(I_{2\times2}\otimes\sigma_+^B)^T]-[A_+(\sigma_-^A\otimes I_{2\times2})\otimes(\sigma_+^A\otimes I_{2\times2})^T\\&+B_+(I_{2\times2}\otimes\sigma_-^B)\otimes(I_{2\times2}\otimes\sigma_+^B)^T]+[A_+(\sigma_+^A\sigma_-^A\otimes I_{2\times2})\\&\otimes I_{4\times4}+B_-(I_{2\times2}\otimes\sigma_+^B\sigma_-^B)\otimes I_{4\times4}]\}\text{Vec}[\overline{\rho_S^{I\prime}(P)}]\\&=\text{Vec}[\rho_S'(0)]. \end{aligned} \quad (25)$$

The 4×4 matrix equation transformed into 16×16 vector equation with the form

$$U(P)_{16\times16}\text{Vec}[\overline{\rho_S^{I\prime}(P)}]=\text{Vec}[\rho_S'(0)], \quad (26)$$

where $\text{Vec}[\overline{\rho_S^{I\prime}(P)}]$ is the vector of row expanding of matrix $\overline{\rho_S^{I\prime}(P)}$. The solution formally is

$$\text{Vec}[\overline{\rho_S^{I\prime}(P)}]=U(P)_{16\times16}^{-1}\text{Vec}[\rho_S'(0)]. \quad (27)$$

Inverse Laplace transformation to time parameter space gives

$$\mathcal{L}^{-1}\text{Vec}[\overline{\rho_S^{I\prime}(P)}]=\mathcal{L}^{-1}U(P)_{16\times16}^{-1}\text{Vec}[\rho_S'(0)], \quad (28)$$

i.e.,

$$\text{Vec}[\rho_S^{I\prime}(t)]=\mathcal{L}^{-1}U(P)_{16\times16}^{-1}\text{Vec}[\rho_S'(0)]. \quad (29)$$

$\mathcal{L}^{-1}U(P)_{16\times16}^{-1}$ can be obtained (see the Appendix) and the master equation is solved. Compared with Markovian approximation, the decoherence rates $\gamma(\omega)$ in our results becomes frequency dependent. Due to the entanglement of two qubits and the interaction with environments, the decay rates for a variety of processes are different, some increase slower, some increase faster, i.e., the decoherence rate and the Lamb shift of the two subsystems $A_\pm$ and $B_\pm$ are renormalized through the entanglement with each other. That is more general and physical.

Therefore, the reduced density matrix $\rho_S'(t)$ in the Schrödinger picture is obtained $\rho_S'(t)=\exp(-iH_0't)\rho_S^{I\prime}(t)\exp(iH_0't)$. The matrix form is

$$\rho_S'(t)=\left[\begin{pmatrix}\exp\left(i\frac{\eta^A\Delta_A}{2}t\right)&0\\0&\exp\left(-i\frac{\eta^A\Delta_A}{2}t\right)\end{pmatrix}\otimes\begin{pmatrix}\exp\left(i\frac{\eta^B\Delta_B}{2}t\right)&0\\0&\exp\left(-i\frac{\eta^B\Delta_B}{2}t\right)\end{pmatrix}\right]\rho_S^{I\prime}(t)\times\left[\begin{pmatrix}\exp\left(-i\frac{\eta^A\Delta_A}{2}t\right)&0\\0&\exp\left(i\frac{\eta^A\Delta_A}{2}t\right)\end{pmatrix}\otimes\begin{pmatrix}\exp\left(-i\frac{\eta^B\Delta_B}{2}t\right)&0\\0&\exp\left(i\frac{\eta^B\Delta_B}{2}t\right)\end{pmatrix}\right]. \quad (30)$$

Transform $\rho_S'(t)$ into $\rho_S(t)$ through $\rho_S(t)=\text{Tr}_R[\exp(-S)\rho_S'(t)\rho_R(0)\exp(S)]$, denoting $X_A=\Sigma_k\frac{g_k}{2\omega_k}\xi_k^A(a_k^+-a_k)$, $X_B=\Sigma_k\frac{f_k}{2\nu_k}\xi_k^B(b_k^+-b_k)$, so

$$\begin{aligned}\rho_S(t)&=\text{Tr}_R[(\cosh X_A-\sinh X_A\sigma_x^A)\otimes(\cosh X_B\\&\quad-\sinh X_B\sigma_x^B)\rho_S'(t)\rho_R(\cosh X_A+\sinh X_A\sigma_x^A)\\&\quad\otimes(\cosh X_B+\sinh X_B\sigma_x^B)]\\&=\frac{1+\eta^A}{2}\frac{1+\eta^B}{2}\rho_S'(t)+\frac{1+\eta^A}{2}\frac{1-\eta^B}{2}(I_{2\times2}\otimes\sigma_x^B)\rho_S'(t)\\&\quad\times(I_{2\times2}\otimes\sigma_x^B)+\frac{1-\eta^A}{2}\frac{1+\eta^B}{2}(\sigma_x^A\otimes I_{2\times2})\rho_S'(t)(\sigma_x^A\end{aligned}$$

$$\otimes I_{2\times 2}) + \frac{1-\eta^A}{2}\frac{1-\eta^B}{2}(\sigma_x^A \otimes \sigma_x^B)\rho_S'(t)(\sigma_x^A \otimes \sigma_x^B). \tag{31}$$

Up to now, we have obtained the reduced density matrix for all kinds of initial states. Although a general solution to this problem, for arbitrary system dynamics and decoherence mechanisms is still out of reach, our technical machinery, developed in the previous section allows treatment of arguably all situations encountered in typical state of the art experiments, such as in quantum optics and condensed matter.

III. THE RESULT AND DISCUSSION

We assume that at $t=0$, the two qubits and environment are described by the state $\rho_{SR}(0)=\exp(-S)\rho_S'(0)\rho_R(0)\exp(S)$ and $\rho_S'(0)\rho_R(0)=\rho_S'(0)\otimes|0_A\rangle|0_B\rangle\langle_B 0|\langle_A 0|$, where $|0_A\rangle|0_B\rangle$ is the vacuum state of two environments. The initial density matrix is prepared only practically coherence of a familiar type (one of the atoms is excited, but it is not certain which one). This is easily expressed in the following form [19,24]:

$$\rho_S'(0) = \frac{1}{3}\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & z & 0 \\ 0 & z^* & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \tag{32}$$

where the factor 1/3 is for notational convenience. The matrix $\rho_S'(0)$ is also spanned in the standard two-qubit product state basis $|1\rangle=|\uparrow\uparrow\rangle,|2\rangle=|\uparrow\downarrow\rangle,|3\rangle=|\downarrow\uparrow\rangle,|4\rangle=|\downarrow\downarrow\rangle$. The basis expansion of matrix is $\rho_S'(0)=a/3|\downarrow\downarrow\rangle\langle\downarrow\downarrow|+(1-a)/3|\uparrow\uparrow\rangle\times\langle\uparrow\uparrow|+1/3(|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle)(\langle\uparrow\downarrow|+\langle\downarrow\uparrow|)$. $a/3$ is the matrix element $\rho_S'(0)_{4,4}$ corresponding to $\langle\downarrow\downarrow|\rho_S'(0)|\downarrow\downarrow\rangle$ and means the A,B qubit are in the excited state, the eigenenergy $\frac{\eta^A\Delta_A}{2}+\frac{\eta^B\Delta_B}{2}$. In order to compare with previous results, consider an important class of mixed state with single parameter a satisfying initially $a\geq 0$, $d=1-a$, and $b=c=z=1$. We will use Wootter's concurrence to quantify the degree of entanglement [25,26]. Let ρ be the density matrix of the pair of qubits expressed in the standard basis. The concurrence may be calculated explicitly from the density matrix ρ for qubits A and B: $C=\max(0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4})$, where the quantities λ_i are the eigenvalues of the matrix M: $M=\rho(\sigma_y^A\otimes\sigma_y^B)\rho^*(\sigma_y^A\otimes\sigma_y^B)$, arranged in decreasing order. Here ρ^* denotes the complex conjugation of ρ in the standard basis. It can be shown that the concurrence varies from 0 for a disentangled state to $C=1$ for a maximally entangled state.

Here and in the following, energies Δ_A, and Δ_B are expressed in units of ω_c, times in units of ω_c^{-1}. For storing entanglement better, we assume two similar qubits and fix $\Delta_A=\Delta_B=0.2$. First consider a very weak qubit-environment interaction $\alpha_A=\alpha_B=0.01$. In Fig. 1, the time evolution of the concurrence for various values of the parameter a is plotted. It shows that for almost all a values between 1/3 and 1, the decay of the concurrence is completed in a finite time, which is the effect of "entanglement sudden death" [15,18] but for smaller a's the time for completed decay is infinite, which is consistent with Refs. [15] and [19]. When the coupling to the environment is weak, we see that the quantum dissipation of the vacuum environment is not sufficient to completely destroy the entanglement in a finite time in some situations. The sudden death of entanglement results from the decays of the mixed double excitation state component. With increasing of the mixed double excitation state component, a value, the concurrence changes from exponential decay to sudden death. The entanglement has another unusual relaxation property: different entanglement states, corresponding to different values of a, with the same initial degrees of entanglement may evolve by different routes, some showing entanglement sudden death, some not, some decaying faster, some slower. That is to say, we can prepare certain initial entanglement states to prolong entanglement time. Most properties of the concurrence in Fig. 1 are consistent with Refs. [19] and [23]. It indicates that in the weak dissipation environment, such as the all-optical setup in Ref. [15], the Markovian approximation and rotating-wave approximation are allowable. From Refs. [19] and [23], the change of decay rate γ, no matter what reason, either from the energy splitting or from qubit-environment interaction strength, does not influence the conversion from the exponential disentanglement to sudden death of entanglement as varying initial state through a. When the interaction strength α become larger, for example in the environment of condensed solid matter, how does the entanglement evolve in non-Markovian processes?

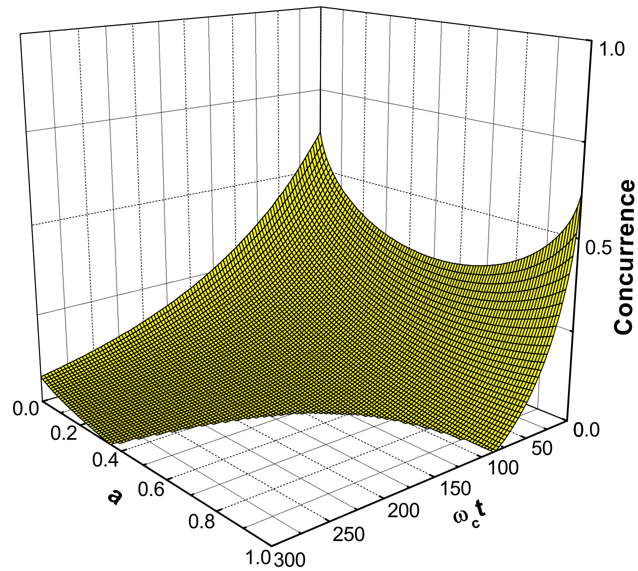

FIG. 1. (Color online) The entanglement decay via spontaneous emission of a pair of two-level qubits starting from the initially entangled state $a/3|\downarrow\downarrow\rangle\langle\downarrow\downarrow|+(1-a)/3|\uparrow\uparrow\rangle\langle\uparrow\uparrow|+1/3(|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle)(\langle\uparrow\downarrow|+\langle\downarrow\uparrow|)$ with a between 0 and 1. the coupling constant of the qubit with the environment $\alpha_A=\alpha_B=0.01$. Here and in the following figures energies Δ_A and Δ_B are expressed in units of ω_c, times in units of ω_c^{-1}. We assume $\Delta_A=\Delta_B=0.2$.

Next, consider large qubit-environment interaction $\alpha_A=\alpha_B=0.05$, the other parameters and initial entanglement state being the same as Fig. 1. The time evolution of the concurrence through the entire range of a values is plotted in

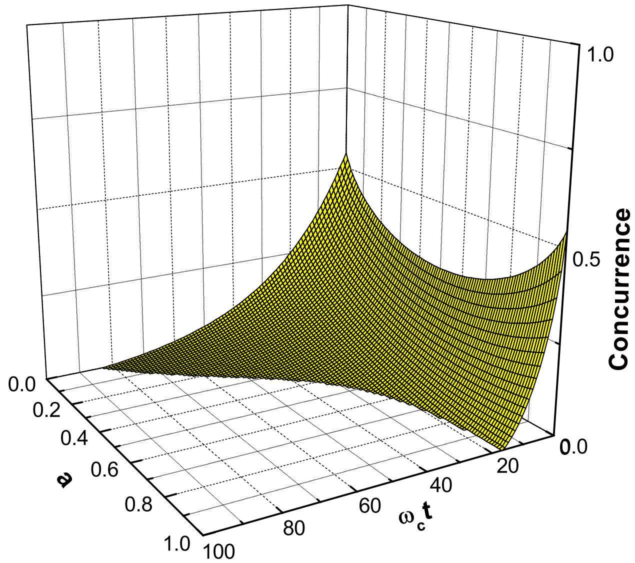

FIG. 2. (Color online) The entanglement decay via spontaneous emission of the qubits with the coupling constants between the qubit and the environment $\alpha_A = \alpha_B = 0.05$. The other parameter the same as Fig. 1.

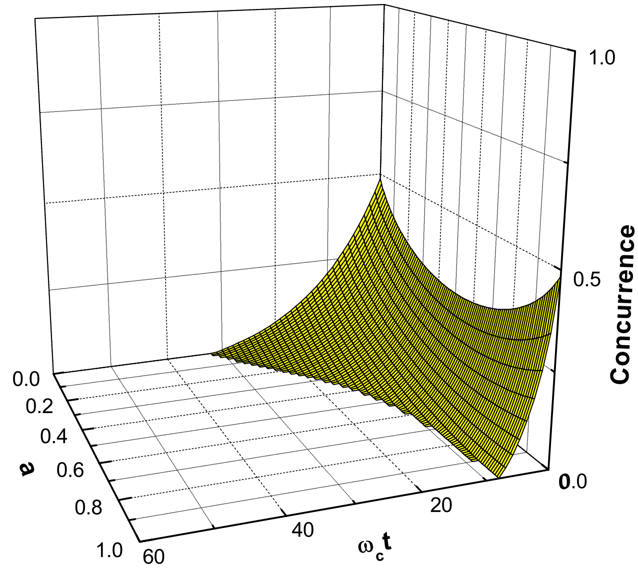

FIG. 3. (Color online) The entanglement decay via spontaneous emission of the pair of two-level qubits. The coupling constants of the qubit to the environment $\alpha_A = \alpha_B = 0.1$. The other parameter the same as Fig. 1.

Fig. 2. As we show, concurrence actually goes abruptly to zero in a finite time and remains zero thereafter. That is to say, the entanglement sudden death always happens in the strong qubit-environment interaction. In Fig. 1, we have shown that the entanglement can last for an infinite period in the vacuum environment for some initial entanglement states. However, in Fig. 2 the sudden death of entanglement always happens no matter which entanglement state the qubits are initially in. So the exponential decay of entanglement is a very special result associated with weak interaction with the dissipation environment at zero temperature. In the disentanglement dynamics of the strong dissipation environment, the lifetime, corresponding to completed death of concurrence, becomes less, and the exponential disentanglement disappears. Furthermore the rotating-wave approximation and Markovian approximation needs to improve, when the coupling to the environment becomes larger, such as in condensed matter systems. This conclusion is also obtained in continuous variable models [28]. Figure 3 shows concurrence for $\alpha_A = \alpha_B = 0.1$, under the same initial condition. It is observed that the lifetime of entanglement decreases with the increasing of the strength of qubit-environment interaction.

IV. CONCLUSION

In this paper, we considered a pair of two-level qubits that are spatially separated from each other and independently coupled to local environments. The dynamics of disentanglement between the qubits is investigated. We show that, for a certain class of two-qubit entangled state, the entanglement measured by concurrence can change from exponential decay to sudden death with the increase of the mixed double excitation state component in the case of weak coupling with the environments. For stronger coupling with the environments, entanglement sudden death always happens no matter what initial entangled state the qubits are in. The exponential decay of entanglement is a very special result for weak dissipative environments. Our results show that the time of ESD depends on the coupling constant α, the energy splittings Δ_A, Δ_B, and cutoff frequency ω_c. Finally, we hope that this work will stimulate more experimental and theoretical works in quantum information and computation.

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APPENDIX

In this appendix, we give details of how to inverse Laplace transformation to time parameter space. $U(P)^{-1}_{16\times16}$ is composed of the matrix elements $\frac{1}{P}$, $\frac{1}{P+A_-}$, $\frac{1}{P+A_+}$, $\frac{1}{P+A_-+A_+}$, $\frac{1}{P+A_-+B_+}$, $\frac{1}{P+A_-+A_++B_+}$, $\frac{1}{P+A_-+A_++B_++B_-}$, etc. Then $\mathcal{L}^{-1}U(P)^{-1}_{16\times16}$ is the inversion of every matrix element. As we know, $\mathcal{L}^{-1}\frac{1}{P}=1$. Solve $\mathcal{L}^{-1}\frac{1}{P+A_-}$, etc., using the following method:

$$\mathcal{L}^{-1}\frac{1}{P+A_-} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp(Pt)}{P + \sum_k \frac{\left(\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}\right)^2}{P - i(\omega_k - \eta^A \Delta_A)}}. \tag{A1}$$

Then changing P to $i\omega + 0^+$ [27],

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(Pt)}{P+\sum_k\frac{\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2}{P-i(\omega_k-\eta^A\Delta_A)}}dP$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-\sum_k\frac{\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2}{(\omega+\eta^A\Delta_A)-\omega_k-i0^+}}d\omega. \quad \text{(A2)}$$

Denote $R(\omega)$ and $\gamma(\omega)$ as the real and imaginary parts of $\sum_k\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2/(\omega-\omega_k-i0^+)$,

$$R(\omega)=\sum_k\wp\frac{\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2}{\omega-\omega_k}$$
$$=(\eta^A\Delta_A)^2\wp\int_0^\infty d\omega'\frac{J(\omega')}{(\omega-\omega')(\omega'+\eta^A\Delta_A)^2}$$
$$=-2\alpha\frac{(\eta^A\Delta_A)^2}{\omega+\eta^A\Delta_A}\left\{\frac{\omega_c}{\omega_c+\eta^A\Delta_A}-\frac{\omega}{\omega+\eta^A\Delta_A}\ln\left[\frac{|\omega|(\omega_c+\eta^A\Delta_A)}{\eta^A\Delta_A(\omega_c-\omega)}\right]\right\} \quad \text{(A3)}$$

and

$$\gamma(\omega)=\pi\sum_k\left(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k}\right)^2\delta(\omega-\omega_k)$$
$$=\pi(\eta^A\Delta_A)^2\frac{J(\omega)}{(\omega+\eta^A\Delta_A)^2}=2\alpha\pi\omega\frac{(\eta^A\Delta_A)^2}{(\omega+\eta^A\Delta_A)^2}, \quad \text{(A4)}$$

where $\wp$ stands for Cauchy principal value

$$\mathcal{L}^{-1}\frac{1}{P+A_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-R(\omega+\eta^A\Delta_A)+i\gamma(\omega+\eta^A\Delta_A)}d\omega$$
$$=\exp[i\omega_{01}t-\gamma(\omega_{01}+\eta^A\Delta_A)t], \quad \text{(A5)}$$

where ω_{01} is the solution of equation $\omega-R(\omega+\eta^A\Delta_A)=0$ and is the Lamb shift due to the local interaction of the qubit A with the environment.

In the same way,

$$\mathcal{L}^{-1}\frac{1}{P+A_+}=\exp[-i\omega_{01}t-\gamma(\omega_{01}+\eta^A\Delta_A)t]. \quad \text{(A6)}$$

It is clear that $\mathcal{L}^{-1}\frac{1}{P+A_-}$ conjugates with $\mathcal{L}^{-1}\frac{1}{P+A_+}$:

$$\mathcal{L}^{-1}\frac{1}{P+A_-+A_+}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega+i\gamma(\eta^A\Delta_A)+i\gamma(\eta^A\Delta_A)}d\omega, \quad \text{(A7)}$$

$$=\exp[-2\gamma(\eta^A\Delta_A)t], \quad \text{(A8)}$$

The decay for $\mathcal{L}^{-1}\frac{1}{P+A_-+A_+}$ accelerated (by a factor of almost 2) compared to $\mathcal{L}^{-1}\frac{1}{P+A_+}$, under the influence of zero temperature environment [17]:

$$\mathcal{L}^{-1}\frac{1}{P+A_-+B_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-R(\omega+\eta^A\Delta_A)-R(\omega+\eta^B\Delta_B)+i\gamma(\omega+\eta^A\Delta_A)+i\gamma(\omega+\eta^B\Delta_B)}d\omega$$
$$=\exp[i\omega_{12}^s t-\gamma(\omega_{12}^s+\eta^A\Delta_A)t-\gamma(\omega_{12}^s+\eta^B\Delta_B)t], \quad \text{(A9)}$$

where ω_{12}^s is the solution of $\omega-R(\omega+\eta^A\Delta_A)-R(\omega+\eta^B\Delta_B)=0$ and is the Lamb shift due to the two environments collective interaction. $\mathcal{L}^{-1}\frac{1}{P+A_++B_+}$ conjugates with $\mathcal{L}^{-1}\frac{1}{P+A_-+B_-}$:

$$\mathcal{L}^{-1}\frac{1}{P+A_++B_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-R(\eta^A\Delta_A-\omega)-R(\omega+\eta^B\Delta_B)+i\gamma(\eta^A\Delta_A-\omega)+i\gamma(\omega+\eta^B\Delta_B)}d\omega$$
$$=\exp[i\omega_{12}^a t-\gamma(\eta^A\Delta_A-\omega_{12}^a)t-\gamma(\omega_{12}^a+\eta^B\Delta_B)t], \quad \text{(A10)}$$

where ω_{12}^a is the solution of $\omega-R(\eta^A\Delta_A-\omega)-R(\omega+\eta^B\Delta_B)=0$ and is also the energy level shift due to the two environments. $\mathcal{L}^{-1}\frac{1}{P+A_-+B_+}$ conjugates with $\mathcal{L}^{-1}\frac{1}{P+A_++B_-}$.

$$\mathcal{L}^{-1}\frac{1}{P+A_++A_-+B_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\exp(i\omega t+0^+)/[\omega-R(\eta^A\Delta_A-\omega)-R(\eta^A\Delta_A+\omega)-R(\omega+\eta^B\Delta_B)+i\gamma(\eta^A\Delta_A-\omega)$$
$$+i\gamma(\eta^A\Delta_A+\omega)+i\gamma(\omega+\eta^B\Delta_B)]d\omega$$
$$=\exp[i\omega_{31}t-\gamma(\eta^A\Delta_A-\omega_{31})t-\gamma(\eta^A\Delta_A+\omega_{31})t-\gamma(\omega_{31}+\eta^B\Delta_B)t], \quad \text{(A11)}$$

where ω_{31} is the solution of $\omega-R(\eta^A\Delta_A-\omega)-R(\eta^A\Delta_A+\omega)-R(\omega+\eta^B\Delta_B)=0$. $\mathcal{L}^{-1}\frac{1}{P+A_++A_-+B_+}$ conjugates with $\mathcal{L}^{-1}\frac{1}{P+A_++A_-+B_-}$:

$$\mathcal{L}^{-1}\frac{1}{P+A_-+B_++B_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\exp(i\omega t+0^+)/[\omega-R(\omega+\eta^A\Delta_A)-R(\eta^B\Delta_B-\omega)-R(\eta^B\Delta_B+\omega)$$
$$+i\gamma(\omega+\eta^A\Delta_A)+i\gamma(\eta^B\Delta_B-\omega)+i\gamma(\eta^B\Delta_B+\omega)]d\omega$$
$$=\exp[i\omega_{32}t-\gamma(\omega_{32}+\eta^A\Delta_A)t-\gamma(\eta^B\Delta_B-\omega_{32})t-\gamma(\eta^B\Delta_B+\omega_{32})t], \quad \text{(A12)}$$

where ω_{32} is the solution of $\omega-R(\omega+\eta^A\Delta_A)-R(\eta^B\Delta_B-\omega)-R(\eta^B\Delta_B+\omega)=0$ and is the energy level shift due to the collective interaction of two environments, too. $\mathcal{L}^{-1}\frac{1}{P+A_-+B_++B_-}$ conjugates with $\mathcal{L}^{-1}\frac{1}{P+A_++B_++B_-}$:

$$\mathcal{L}^{-1}\frac{1}{P+A_++A_-+B_++B_-}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega+i\gamma(\eta^A\Delta_A-\omega)+i\gamma(\eta^A\Delta_A+\omega)+i\gamma(\omega+\eta^B\Delta_B)+i\gamma(\eta^B\Delta_B-\omega)}d\omega$$
$$=\exp[-2\gamma(\eta^A\Delta_A)t-2\gamma(\eta^B\Delta_B)t]. \quad \text{(A13)}$$

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