Non-Markovian dynamical effects and time evolution of the entanglement entropy of a dissipative two-state system

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Abstract – We investigate the dynamical information exchange between a two-state system and its environment which is measured by the von Neumann entropy. It is found that in the underdamping regime, the entropy dynamics exhibits an extremely non-Markovian oscillation-hump feature, in which oscillations manifest quantum coherence and a hump of envelop demonstrates temporal memory of bath. It indicates that the process of entropy exchange is bidirectional. When the coupling strength increases up to a certain threshold, the hump along with ripple disappears, which is indicative of the coherent-incoherent dynamical crossover. The long-time limit of entropy evolution reaches the ground-state value which agrees with that of the numerical renormalization group.

The non-equilibrium evolution of open quantum systems is one of the most challenging and intriguing problems of contemporary research in both theoretical and experimental physics. The transient dynamics can be harnessed and controlled desirably in quantum information processing. The first step to manipulate it is to understand how it evolves in a short-time interval. It is known that the correlations of an open system with its surrounding environment lead to finite lifetime of quantum superpositions, which give rise to the evolution from pure states into mixed ones. It is often stated that decoherence causes the system to become entangled with its environment, and the entanglement between them can be measured quantitatively by the von Neumann entropy [1]. The main questions which now arise are: How does the entropy or quantum information flow from system to environment? After the state of the system is initialized as a pure state without entanglement, how does the entropy evolve to its long-time limit (monotonously or not)? Is the process of information transfer between bath and system, unidirectional or bidirectional? In this paper, as far as we know, it is the first time that one shows the time evolution of entropy for an open system which exhibits extremely non-Markovian characters and points out that the process of entropy exchange is bidirectional in the underdamping region.

The dissipative two-state system (TSS), which is also called the spin-boson model, as a simple paradigm of open system, is a generic model which can be widely used to describe a large number of physical and chemical processes, such as the defect-tunneling and electron transfer, and applied to clarify very interesting quantum phenomena, such as decoherence and dephasing [2,3]. The open system inevitably encounters decoherence which makes a quantum superposition state decay into a classical, statistical mixture of states. Thus, derivation of the reduced density matrix is a central goal in order to describe its evolution. Based on the weak-coupling assumption, a Markovian master equation could give its dynamics. However, strong interactions with low-temperature reservoirs give rise to large system-environment correlations which generally result in a failure of the Markovian approximation. In this case, the system dynamics possesses long memory times and exhibits a pronounced non-Markovian behavior [4]. Thus, it is significant to show the temporary evolution of the system by a non-Markovian approach, especially, in the case of the strong coupling to its bath. The non-Markovian approach can investigate not only a more complicated situation where Markovian approximation is unreachable but also different spectral densities between the system and the environment.

The entanglement entropy has been considered in many works. They mainly focus on the static or ground-state

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properties and find some important results [5–7]. Costi and McKenzie used a numerical renormalization group (NRG) treatment to study the entropy of the ground state as a function of coupling [5]. Recently, Hur and coworkers applied the NRG to study the quantum phase transition and found that there is a cusp in the entanglement entropy accompanying quantum phase transition [6]. As explicitly pointed out by Costi and McKenzie, it is fascinating to show how entropy varies with time after the qubit is initially prepared in a certain state without entanglement [5]. In order to calculate the dynamics of entanglement entropy, we present an analytical approach based on a unitary transformation method without the Markovian approximation. It works well in the parameter regime $0 < \alpha < 1$ and $0 < \Delta < \omega_c$, and can reproduce well-known non-perturbation results obtained by various methods, such as the coherence-incoherence transition, which have been studied in our previous work [8]. The approach does not invoke the rotating-wave approximation so as to take into account the effects of counter-rotating terms on transient dynamics. Since the quantum manipulation can be effectively made in the coherent region, we would give the evolution of entropy for this regime at $T = 0$. We find the non-Markovian entropy evolution with a pronounced small-oscillations feature in the weak coupling, which demonstrates quantum coherence. As the coupling increases, a hump along with ripple clearly emerges in the short time characterizing the temporal memory of the bath, eventually the dissipative effects quench some oscillations and the hump near the coherence-incoherence transition.

The spin-boson model reads [2,3]

$$H = -\frac{\Delta}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k + \frac{1}{2}\sum_k g_k(b_k^\dagger + b_k)\sigma_z. \quad (1)$$

Standard notations are used [3], Δ is the bare tunneling matrix and g_k the coupling constant. The effect of environment is determined by its spectral density: $\sum_k g_k^2\delta(\omega - \omega_k) = 2\alpha\omega\theta(\omega_c - \omega)$, where α is the dimensionless coupling constant, ω_c is a cutoff frequency and $\theta(x)$ is the usual step function. (In the work the spectrum is of Ohmic type, and we set $\hbar = k_B = 1$.) Although the model seems quite simple, it is in general not exactly solvable and a large variety of approximate analytical and numerical methods have been proposed and implemented to study its ground state and dynamics [5–15].

Unitary transformation. – A unitary transformation, which is defined as $H' = \exp(S)H\exp(-S)$, is applied to H in order to take into account the correlation between the spin and bosons [8,9]. The form of the generator is proposed,

$$S = \sum_k \frac{g_k}{2\omega_k}\xi_k(b_k^\dagger - b_k)\sigma_z, \quad (2)$$

where a k-dependent function ξ_k is introduced [8]. The transformation can be performed to the end and the result

is $H' = H_0' + H_1' + H_2'$,

$$H_0' = -\frac{\Delta_r}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k - \sum_k \frac{g_k^2}{4\omega_k}\xi_k(2 - \xi_k), \quad (3)$$

$$H_1' = \frac{1}{2}\sum_k g_k(1 - \xi_k)(b_k^\dagger + b_k)\sigma_z - \frac{\Delta_r}{2}i\sigma_y B, \quad (4)$$

$$H_2' = -\frac{\Delta}{2}\sigma_x(\cosh\{B\} - \eta) - \frac{\Delta}{2}i\sigma_y(\sinh\{B\} - \eta B), \quad (5)$$

where $B = \sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)$ and $\Delta_r = \eta\Delta$. The renormalized factor of tunneling is $\eta = \exp[-\sum_k \frac{g_k^2}{2\omega_k^2}\xi_k^2]$. H_0' is the unperturbed part of H' and, obviously, it can be solved exactly since the spin and bosons are decoupled. The ground state of H_0' is $|g_0\rangle = |s_1\rangle|\{0_k\}\rangle$ ($\sigma_x|s_1\rangle = |s_1\rangle$), $|\{0_k\}\rangle$ is the vacuum state for every boson mode $n_k = 0$). H_1' and H_2' are treated as perturbation and they should be as small as possible. For this purpose η is determined to make $\text{Tr}_B(\rho_B H_2') = 0$, where ρ_B is the density operator of bath. Besides, ξ_k is determined as

$$\xi_k = \frac{\omega_k}{\omega_k + \Delta_r}, \quad (6)$$

and because of this form H_1' is rewritten as

$$H_1' = \sum_k V_k\left[b_k^\dagger\sigma_- + b_k\sigma_+\right], \quad (7)$$

where $V_k = \Delta_r g_k\xi_k/\omega_k$ and $\sigma_- = (\sigma_z - i\sigma_y)/2$, $\sigma_+ = (\sigma_z + i\sigma_y)/2$. When $T = 0$ it is easy to check that $H_1'|g_0\rangle = 0$. This is essential in our approach.

In our treatment H_0' is treated as the unperturbed Hamiltonian, in which the tunneling has been already renormalized by η coming from the contribution of diagonal transition of bosons. H_1' is the perturbation relating to the non-diagonal transition of single-boson, and H_2', containing all other multi-boson non-diagonal transitions, is omitted because its contribution to physical quantities is $O(g_k^4)$ and higher. Note that $0 \leqslant \xi_k \leqslant 1$. ξ_k measures the adiabatic intensity of the particle interacting with its environment [8]. $\xi_k \sim 1$ if $\omega_k \gg \Delta_r$, while $\xi_k \ll 1$ for $\omega_k \ll \Delta_r$. In addition, by the choice of ξ_k, H_1' has taken into account the effects of counter-rotating terms. In other words, the bare coupling $g_k/2$ in the original Hamiltonian is replaced by the renormalized coupling V_k after the unitary transformation.

Density operator. – In order to show the quantum dynamics, we would first give the density operator in Schrödinger representation, $\rho_{SB}(t)$ with Hamiltonian H, where the subscript SB stands for the spin-boson model. For the transformed Hamiltonian H' the density operator is $\rho_{SB}'(t) = e^S\rho_{SB}(t)e^{-S}$. The density operator in the interaction representation is

$\rho_{SB}^{\prime I}(t) = \exp(iH_0't)\rho_{SB}'(t)\exp(-iH_0't)$. By the equation of motion for $\rho_{SB}^{\prime I}(t)$ [16], we obtain the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S^{\prime I}(t) = -\int_0^t \mathrm{Tr}_B[H_1'(t),[H_1'(t'),\rho_S^{\prime I}(t')\rho_B]]\,\mathrm{d}t', \quad (8)$$

where $\rho_S^{\prime I}(t) = \mathrm{Tr}_B\rho_{SB}^{\prime I}(t)$ and $H_1'(t) = \exp(iH_0't)H_1' \exp(-iH_0't)$. It is known that one can arrive at the Born-Markov approximation equation neglecting retardation in the integration, i.e., $\rho_S^{\prime I}(t')$ is replaced by $\rho_S^{\prime I}(t)$. Our treatment is beyond this approximation.

At $t = 0$, the usual initial density operator is $\rho_{SB}(0) = \rho_S(0)\rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\rho_B$. Then we can get the initial condition for our calculations: $\rho_{SB}^{\prime I}(0) = \rho_{SB}'(0) = e^S\rho_{SB}(0)e^{-S}$ leads to $\rho_S^{\prime I}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The calculation is up to the second order g_k^2 and the details are shown in the Appendix. The solution of reduced density operator $\rho_S'(t) = \begin{pmatrix} \rho_{11}' & \rho_{12}' \\ \rho_{21}' & \rho_{22}' \end{pmatrix}$ is

$$\rho_{11}'(t) - \rho_{22}'(t) = \frac{1}{4\pi i}\int_\infty^{-\infty} e^{-i\omega t}\mathrm{d}\omega F^*(\omega)$$
$$+ \frac{1}{4\pi i}\int_{-\infty}^\infty e^{i\omega t}\mathrm{d}\omega F(\omega), \quad (9)$$

$$\rho_{12}'(t) + \rho_{21}'(t) = 1 - \frac{1}{2\pi i}$$
$$\times \int_{-\infty}^\infty \frac{e^{i\omega t}\mathrm{d}\omega}{\omega - \sum_k\left[\frac{V_k^2}{\omega + \omega_k - \Delta_r - i0^+} + \frac{V_k^2}{\omega - \omega_k + \Delta_r - i0^+}\right]}, \quad (10)$$

where $F(\omega) = (\omega - \Delta_r - \sum_k \frac{V_k^2}{\omega - \omega_k - i0^+})^{-1}$. The real and imaginary parts of $\sum_k V_k^2/(\omega - i0^+ - \omega_k)$ are denoted as

$$R(\omega) = -2\alpha\frac{\Delta_r^2}{\omega + \Delta_r}$$
$$\times \left\{\frac{\omega_c}{\omega_c + \Delta_r} - \frac{\omega}{\omega + \Delta_r}\ln\left[\frac{\omega(\omega_c + \Delta_r)}{\Delta_r(\omega_c - \omega)}\right]\right\}, \quad (11)$$

$$\gamma(\omega) = 2\alpha\pi\omega\frac{\Delta_r^2}{(\omega + \Delta_r)^2}\,(0 \leqslant \omega \leqslant \omega_c), \quad (12)$$

respectively.

Dynamical quantities. – In what follows we calculate the dynamical quantities, $\langle\sigma_\tau(t)\rangle = \mathrm{Tr}_S\mathrm{Tr}_B[\rho_{SB}\sigma_\tau](\tau = x, y, z)$. The reduced density operator of the original Hamiltonian H is $\rho_S(t) = \mathrm{Tr}_B\rho_{SB}(t)$, which can be expressed as $\rho_S(t) = \frac{1}{2}[1 + \sum_\tau \langle\sigma_\tau(t)\rangle\sigma_\tau]$. First, we calculate $\langle\sigma_z(t)\rangle$ which is usually denoted as $P(t)$ in the literature,

$$P(t) = \mathrm{Tr}_S\mathrm{Tr}_B(\rho_{SB}(t)\sigma_z) =$$
$$\frac{1}{\pi}\int_0^{\omega_c}\mathrm{d}\omega\frac{\gamma(\omega)\cos(\omega t)}{[\omega - \Delta_r - R(\omega)]^2 + \gamma^2(\omega)}, \quad (13)$$

since $\mathrm{Tr}_B\rho_B = 1$. The integration in eq. (9) can be done approximately by the residue theorem, $P(t) = \cos(\omega_0 t)\exp(-\gamma t)$, where ω_0 is the solution of equation $\omega_0 - \Delta_r - R(\omega_0) = 0$, and γ is the Wigner-Weisskopf approximation of $\gamma(\omega)$: $\gamma = \frac{\pi}{2}\alpha\Delta_r$. The solution ω_0 is real only when $\alpha < \alpha_c$, $\alpha_c = (1 + \Delta_r/\omega_c)/2$. This becomes the well-known result $\alpha_c = 1/2$ in the scaling limit $\Delta/\omega_c \ll 1$ [2,3]. For $\alpha > \alpha_c$ there is no real solution ω_0 and it means that $\alpha = \alpha_c$ determines the critical point corresponding to the coherent-incoherent transition. The coherent regime $\alpha < \alpha_c$ can be divided into the underdamping part and the overdamping one by a criterion $\omega_0 > \gamma(\omega_0)$ (underdamping), or $\omega_0 < \gamma(\omega_0)$ (overdamping). In the scaling limit $\Delta/\omega_c \ll 1$ the point where $\omega_0 = \gamma(\omega_0)$ is at $\alpha_c^* = 0.325$, which is very close to the previous results $\alpha = 1/3$ or 0.3 [12,13]. From eq. (1) one can get a relation between $\langle\sigma_y(t)\rangle$ and $\langle\sigma_z(t)\rangle$, $\langle\sigma_y(t)\rangle = -\frac{1}{\Delta}\frac{d}{dt}\langle\sigma_z(t)\rangle$, since $i[H,\sigma_z] = -\Delta\sigma_y$. $\langle\sigma_x(t)\rangle$ can be calculated in the following:

$$\langle\sigma_x(t)\rangle = \mathrm{Tr}_S\mathrm{Tr}_B(\rho_{SB}'(t)e^S\sigma_x e^{-S}) =$$
$$\eta\left\{1 - \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Gamma(\omega)\cos(\omega t)\,\mathrm{d}\omega}{[\omega - \Sigma(\omega)]^2 + \Gamma^2(\omega)}\right\}, \quad (14)$$

where $\Sigma(\omega) = R(\Delta_r + \omega) - R(\Delta_r - \omega)$ and $\Gamma(\omega) = \gamma(\Delta_r + \omega) + \gamma(\Delta_r - \omega)$. One can check that the initial conditions $\langle\sigma_x(0)\rangle = 0, \langle\sigma_y(0)\rangle = 0, \langle\sigma_z(0)\rangle = 1$ are exactly satisfied. Besides, $\langle\sigma_x(\infty)\rangle = \eta, \langle\sigma_y(\infty)\rangle = 0, \langle\sigma_z(\infty)\rangle = 0$, which are the correct results for thermodynamic equilibrium state [5].

Entropy of entanglement. – The entropy (indeterminacy of the state) is a measure of the missing information compared with the pure state of the composite system. The more information about the composite state is lost, the more information is contained in the correlation between the substates. The greater the entropy of system, the more strongly is the pure state of the composite system correlated and thus entangled [17]. To see what happens to the coherence properties due to the interaction between the system and its surrounding starting from a pure state, we use the von Neumann entropy. It is defined as $S(t) = -\mathrm{Tr}(\rho_S\log_2\rho_S)$, which is a measure of the entanglement between them. It may be expressed in terms of the eigenvalues $\lambda_\pm(t) = 1/2 \pm \sqrt{\langle\sigma_x(t)\rangle^2 + \langle\sigma_y(t)\rangle^2 + \langle\sigma_z(t)\rangle^2}/2$ of the density operator ρ_S as, $S(t) = -\lambda_+\log_2\lambda_+ - \lambda_-\log_2\lambda_-$.

From the Hamiltonian (eq. (1)), it predicts that $\langle\sigma_y(\infty)\rangle = 0, \langle\sigma_z(\infty)\rangle = 0$, and only $\langle\sigma_x(\infty)\rangle$ is nonvanishing in the delocalized phase, which verify our obtained results. So, the entropy in the long-time limit S_{eq} is given by $\lambda_\pm = 1/2 \pm \langle\sigma_x\rangle/2$, which is shown in fig. 1 along with the NRG results [5]. As α increases, S_{eq} becomes large. When $\alpha \to 1$, S_{eq} tends to one. In the scaling limit, for $\alpha > 1$, $\langle\sigma_x\rangle = 0$ and the system remains its initial state, thus $|\langle\sigma_z\rangle| = 1$ and $S_{eq} = 0$. In other words, the transition between the localized and the delocalized phase occurs
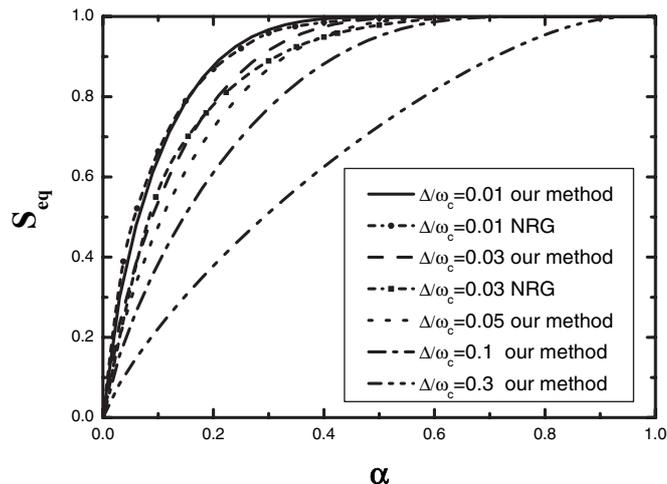
Fig. 1: S_{eq} as a function of α for fixed $\Delta/\omega_c = 0.01$, 0.03, 0.05, 0.1, and 0.3. The NRG results is shown for comparison.

at $\alpha = 1$ and the entropy decreases from unity to zero abruptly.

In order to calculate the entropy, $\langle\sigma_x(\infty)\rangle$ can also be evaluated using the NRG applied to the equivalent anisotropic Kondo model. The NRG data shown in fig. 1 are taken from ref. [5]. It is seen that for small tunneling our result is in good agreement with those of NRG. However, with increasing large tunneling some discrepancies appear for moderate values of the coupling. We think that it comes possibly from the NRG discretization [13].

The dynamics of entanglement entropy displays extremely non-Markovian features. Figure 2 shows $S(t)$ for different couplings with $\Delta = 0.1\omega_c$. For $\alpha < \alpha_c^*$, the entropy increases non-monotonically from zero to a finite value (S_{eq}) with explicit oscillations, and would not come close to saturation in the short-time interval which means that quantum coherence is not directly destroyed by the bath. At the same time, the envelope of entropy exhibits a hump characterizing the short-time memory of the bath. On the other hand, the oscillation-hump feature demonstrates that the process of entropy exchange is bidirectional. To better understand the nature of the oscillation-hump feature and the large contribution from quantum fluctuations, we should consider the elements of the reduced density matrix. $\langle\sigma_y(t)\rangle$ and $\langle\sigma_z(t)\rangle$ exhibit oscillations which represent coherence. So, the dominant contribution to the oscillatory signal comes from $\langle\sigma_y(t)\rangle^2 + \langle\sigma_z(t)\rangle^2$, while the trend of entropy evolution ascribes to $\langle\sigma_x(t)\rangle$. Thus, the oscillation of entropy shows the coherent evolution in coherent regime.

As coupling increases, oscillations become obviously weaker with small amplitudes and the envelop of entropy rises rapidly with small hump (see fig. 2(b)) due to the effects of strong dissipation. Near the crossover from coherent to incoherent regime $\alpha \sim 1/2$, the entropy shape displays faster rising behaviors without oscillation and the hump disappears, which is an important character

corresponding to the coherent-incoherent crossover. Note that S_{eq} is analytic and continuous at α_c because no phase transition happens at this point while the dynamical crossover from damped oscillatory to pure decaying behaviors takes place. Thus, we cannot extract a distinguishable feature of this crossover from S_{eq} because of its character of thermodynamical equilibrium even if $\alpha > \alpha_c$ S_{eq} is near to its saturation. Therefore, only transitory dynamics of entropy could give the indicator of the crossover even though S_{eq} might also be regarded as an interesting order parameter to mark quantum phase transitions.

The evolution of entanglement entropy is very different from that of Markovian approximation. The dynamics of $S(t)$ is shown in fig. 3(a) with several tunnelings for $\alpha = 0.2$ as well as the corresponding Markovian results. In the Markovian evolution, the system undergoes a smooth and fast relaxation to its final statistical mixture. It is found that there is no short-time oscillations in the Markovian evolution. Thus, the transient oscillatory behaviors of entropy dynamics cannot be correctly described by the Markovian approximation. Nevertheless, in the long-time limit, Markovian results are consistent with S_{eq} as expected. The oscillation of the entropy is a hallmark of non-Markovian dynamics in the coherent regime which is unexpected in the Markovian dynamics. From the scaled entropy $S(t)/S_{eq}$ in fig. 3(b), one can see that the entropy displays almost synchronously with different amplitudes of oscillations for any tunneling and eventually goes wiggly down to S_{eq}. It indicates that the system exchanges entropy frequently with its environment in the short time. In this case, the oscillations are more pronounced for the enhancement of coherence-involved transition between two states. From another point of view, the ability of exchanging information becomes strong for the system with increasing tunneling and it remains coherent for a longer time. (Note that the unit of time is $\eta\Delta$, which becomes explicitly larger with increasing tunneling.)

In the coherent regime, a sufficient number of quantum manipulations can be performed within the coherent time. The need to maintain quantum coherence during the operation is especially difficult to achieve in solid-state systems such as quantum dots which couple relatively strongly to uncontrollable environmental degrees of freedom, leading to decoherence. Only in the underdamping regime, the quantum control has more efficiency. The promising experimental proposal that entanglement entropy can be measured in Cooper pair box or quantum dot scheme is suggested by Kopp and Le Hur recently [6]. We really expect that experimental setup is capable of testing our predictions and such measurements would provide a proof of the existence of oscillations in the entropy evolution although it is not easy to probe small signals in the background of noises and thermal fluctuations.

Summary. – The entanglement entropy dynamics of dissipative TSS is studied by means of the analytical approach on the basis of a unitary transformation.
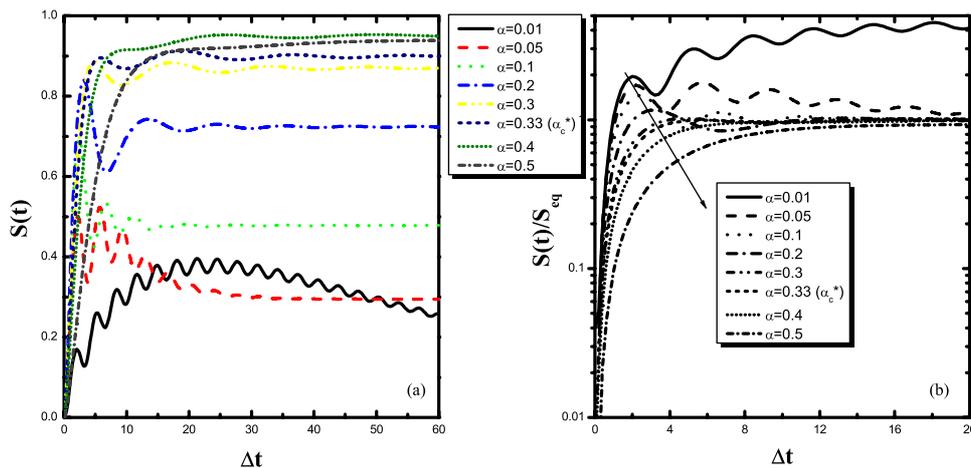
Fig. 2: (Colour on-line) (a) Time evolution of entanglement entropy $S(t)$ for $\Delta/\omega_c = 0.1$ with different couplings (the underdamping-overdamping transition point is $\alpha_c^* = 0.33$). (b) The scaled entropy $S(t)/S_{\text{eq}}$ is also shown. The arrow indicates the increasing tendency of coupling.
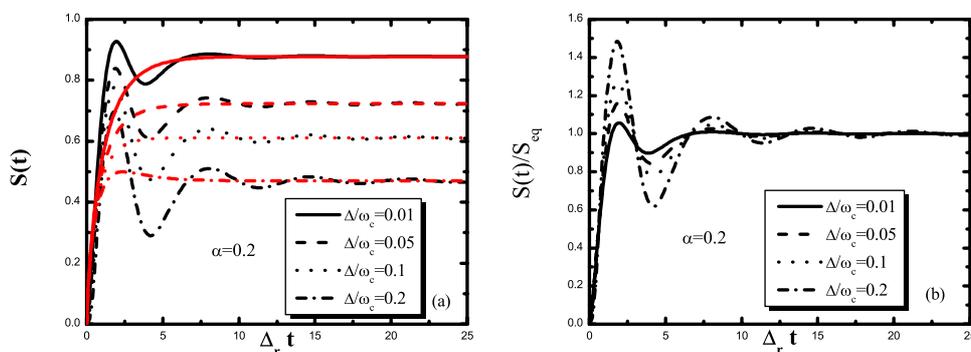
Fig. 3: (Colour on-line) (a) Time evolution of $S(t)$ as a function of $\Delta_r t$ for fixed $\alpha = 0.2$ with different tunnelings $\Delta/\omega_c = 0.01$, 0.05, 0.1 and 0.2 (black curves) along with corresponding Markovian results (smooth red lines without wiggly oscillations). (b) The scaled entropy $S(t)/S_{\text{eq}}$ is shown for its synchronization.

Analytical results of the quantum dynamics, described by the $\rho_S(t)$, is obtained for the general finite Δ/ω_c case. The entanglement entropy evolution from a pure state is shown with explicit non-Markovian features. Our approach is quite simple and tractable without spectral structure dependence, and it could trigger many future applications in other more complicated coupling systems with realistic spectrum function, such as superconducting qubit with Lorentz spectrum.

Here are a few words about the key ingredient of the approach. The purpose of our unitary transformation is to find a better way to divide the transformed Hamiltonian into unperturbed part H_0', which can be treated exactly, and perturbation ones $H_1' + H_2'$, which may be treated by perturbation theory. In H_0' the tunnelling has been already renormalized by η which comes from the contribution of diagonal transition of bosons. H_1' is related to the non-diagonal transition of single-boson and all other multi-boson non-diagonal transitions are contained in H_2'. If one treats the coupling term in the original Hamiltonian H as the perturbation, the dimensionless expanding parameter is g_k^2/ω_k^2. For Ohmic bath $s = 1$ it is $2\alpha/\omega$ which is logarithmic divergent in the infrared limit. By choosing the form of η and introducing the function ξ_k in the unitary transformation it is possible to treat H_1' and H_2' as perturbation because of the following reason. On account of the form of η H_2' can be treated as perturbation because its contribution is zero at second order of g_k. The effect of the coupling term in H' (H_1') can be safely treated by perturbation theory because the infrared divergence in the original perturbation treatment for H is eliminated by making choice of the function form ξ_k. The expanding parameter ($s = 1$) is $g_l^2 \xi_l^2/\omega_l^2 \sim 2\alpha\omega/(\omega + \eta\Delta)^2$, which is finite in the infrared limit. Besides, our approach is well checked not only by the initial values of the correlation functions and entanglement entropy, such as $P(t = 0) = 0$, $S(t = 0) = 0$, and their long-time limits such as $P(\infty) = 0$, $S(\infty) = S_{eq}$.

APPENDIX

In this appendix we list the details of solving the master equation (8). The integration in eq. (8) can be done as follows:

$$
-\int_0^t \mathrm{Tr}_B[H_1'(t),[H_1'(t'),\rho_S'^I(t')\rho_B]]\,\mathrm{d}t' =
$$

$$
\begin{aligned}
-\sum_k V_k^2 \int_0^t \mathrm{d}t'\big\{ &[n_k\sigma_-\sigma_+\rho_S'^I(t') - (n_k+1)\sigma_-\rho_S'^I(t')\sigma_+ \\
&-n_k\sigma_+\rho_S'^I(t')\sigma_- + (n_k+1)\rho_S'^I(t')\sigma_+\sigma_-] \\
&\times \exp[i(\omega_k-\Delta_r)(t-t')] + [(n_k+1)\sigma_+\sigma_-\rho_S'^I(t') \\
&-n_k\sigma_+\rho_S'^I(t')\sigma_- - (n_k+1)\sigma_-\rho_S'^I(t')\sigma_+ \\
&+n_k\rho_S'^I(t')\sigma_-\sigma_+]\exp[-i(\omega_k-\Delta_r)(t-t')]\big\},
\end{aligned}
\tag{A.1}
$$

where $n_k = 1/[\exp(\beta\omega_k)-1]$ is the Bose function. Thus, eq. (8) can be solved by the Laplace transformation. If we denote

$$
\rho_S'^I(p) = \begin{pmatrix} \rho_{11}'^I & \rho_{12}'^I \\ \rho_{21}'^I & \rho_{22}'^I \end{pmatrix},
$$

the solution of eq. (A.1) is

$$
\begin{aligned}
\rho_{11}'^I - \rho_{22}'^I = {} & \frac{1/2}{p + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p + i(\omega_k - \Delta_r)}} \\
& + \frac{1/2}{p + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p - i(\omega_k - \Delta_r)}},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\rho_{12}'^I - \rho_{21}'^I = {} & \frac{1/2}{p + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p + i(\omega_k - \Delta_r)}} \\
& - \frac{1/2}{p + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p - i(\omega_k - \Delta_r)}},
\end{aligned}
\tag{A.3}
$$

$$
\rho_{12}'^I + \rho_{21}'^I = \frac{\sum_k \dfrac{2V_k^2}{p^2 + (\omega_k - \Delta_r)^2}}{p\left(1 + 2\sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p^2 + (\omega_k - \Delta_r)^2}\right)}.
\tag{A.4}
$$

Using the relation between the Schrödinger and the interaction representation and making the Laplace inverse-transformation, we can get

$$
\begin{aligned}
&\rho_{11}'(t) - \rho_{22}'(t) = \cos(\Delta_r t)(\rho_{11}'^I(t) - \rho_{22}'^I(t)) \\
&- i\sin(\Delta_r t)(\rho_{12}'^I(t) - \rho_{21}'^I(t)) = \\
&\frac{1}{4\pi i}\int e^{pt}\mathrm{d}p\Bigg\{ \frac{1}{p + i\Delta_r + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p + i\omega_k}} \\
&+ \frac{1}{p - i\Delta_r + \sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p - i\omega_k}} \Bigg\},
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
&\rho_{12}'(t) + \rho_{21}'(t) = \rho_{12}'^I(t) + \rho_{21}'^I(t) = \\
&\frac{1}{2\pi i}\int e^{pt}\mathrm{d}p \frac{\sum_k V_k^2 \dfrac{2}{p^2 + (\omega_k - \eta\Delta)^2}}{p\left(1 + 2\sum_k \dfrac{V_k^2 \coth(\omega_k/2T)}{p^2 + (\omega_k - \eta\Delta)^2}\right)}.
\end{aligned}
\tag{A.6}
$$

The integration path is on a line parallel to the imaginary axis of the complex p-plane from $p = 0^+ - i\infty$ to $p = 0^+ + i\infty$.

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