Non-Markovian disentanglement dynamics of two-qubit system in common environment

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Abstract. The disentanglement of a pair of identical qubits, sharing a dissipative environment, is investigated. We proposed a new non-Markovian analytical method to study the model, which takes into account the counter-rotating term in the qubit-environment interaction. It is shown that when the qubits sufficiently separate from each other, the model is equivalent to each subsystem interacts independently with its local environment. The effects of the dissipative environment and the interqubit distance on the time evolution of the concurrence for four bell states are examined in detail. And point out some new features appearing in the case of the two qubits close together.

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1 Introduction

The role of quantum entanglement is of primary importance in quantum information [1,2] and computation [3,4]. In recent years a lot of researches have been devoted to study the disentanglement dynamics of two subsystems in various environments [5–12]. For example, Almeida et al. [12] showed that, using an all-optical experimental setup, even when the environment-induced decay of each system is asymptotic, quantum entanglement may appear “entanglement sudden death”, called ESD. Among these papers, the presence of an environment, e.g. a generic noisy reservoir or a heat bath, is commonly thought as counteracting entanglement holding or creation. While Benatti et al. [13] showed that two noninteracting two-level systems, immersed in a common bath, can become mutually entangle when evolving according to a Markovian dynamics. Braun et al. draw the same conclusion [14,15]. In this sense, except to destroy the entanglement, sometimes environment is in favor of entanglement. It has also been recognized that the collective properties of multi-atom systems can alter spontaneous emission compared with the single atom case. As it was firstly pointed out by Dicke, the interaction between the atomic dipoles could cause the multiatom system to decay with two significantly different, one enhanced and the other reduced, spontaneous emission rates. We want to know the effect of the common dissipative environment on evolution of entanglement in solid qubits, constructive or destructive? what does it depend on?

In this paper we examined the non-Markovian disentanglement dynamics of two qubits in common bosonic environment, without rotating-wave approximation in the interaction of qubit-environment. Firstly, apply an unitary transformation on the Hamiltonian, then use perturbation approach on the transformed Hamiltonian. Therefore, the interaction between system-environment in the transformed Hamiltonian is of a similar rotating-wave (energy conservation) form, that is important for posterior convolution operation. An analytical expression of density matrix for random original state and various environment is obtained in the master equation. Considering four Bell states as initial states, we find that the appearance of entanglement sudden death strongly depends on the original state and the coupling strength to the environment. We also find that the separation between two qubits does not always have a detrimental effect on the entanglement of quantum system. In summary, the dynamics of the quantum entanglement is sensitive to the initial states, the distance between two qubits, the strength of qubit-environment interaction.

The paper is organized as follows: in Section 2 we introduce the Hamiltonian of the model and solve it in terms of non-Markovian treatment. For four Bell states as initial states, the dependence of the concurrence on the separation between two qubits and the coupling strength to the dissipative environment, are discussed in Section 3. Finally, the conclusion is given in Section 4.

2 The model and theory

This paper is concerned a pair of two-level systems, sharing a environment, since it is generally believed that...
entanglement of only two microscopic quantum systems (qubits, atoms) is essential to implement quantum protocols such as quantum computation. Within the last decade, various schemes for quantum computation have been proposed and many of them have even been realized, such as superconductive flux qubit and solid quantum dot qubit. We denote the subsystem in the Hamiltonian as qubit for convenience. Each of the two qubits is a two-level system that is described in pseudospin notation. The model may be formulated to the following Hamiltonian (set $\hbar = 1$):

$$H = H_{qu} + H_{env} + H_{int},$$

with

$$H_{qu} = -\frac{1}{2} \sum_j \Delta_j \sigma_{jz},$$

$$H_{env} = \sum_k \omega_k b_k^+ b_k,$$

$$H_{int} = \frac{1}{2} \sum_{j,k} \left[ g_k(r_j)b_k^+ + g_k(r_j)b_k \right] \sigma_{jx},$$

where the Hamiltonian of the two qubits is $H_{qu}$, that of the environment is $H_{env}$. $H_{int}$ is the interaction between qubits and environment, without rotating wave approximation. Here $\sigma_x, \sigma_z$ denotes the usual Pauli spin matrices, $\Delta_j$ ($j = 1, 2$) describes the energy splitting in qubit $j$, $b_k^+$ ($b_k$) and $\omega_k$ are the creation (annihilation) operator and energy with wave vector $k$ in the qubits’ environment. $g_k(r)$ is the qubit-environment coupling strength. Our approach is not restricted by the form of environment or spectral density. In general, the spatial dependence of the interaction $g_k(r)$ use the form $g_k(r_j) = g_k \exp[i \vec{k} \cdot \vec{r_j}]$ [15]. The spectral density of the environment is defined by:

$$J_{11}(\omega) = \int \frac{dk}{2\pi} [g_k(r_1)]^2 \delta(\omega - \omega_k)$$

$$= \sum_k g_k^2 \delta(\omega - \omega_k),$$

$$J_{12}(\omega) = \int \frac{dk}{2\pi} [g_k(r_1)] \omega_k \delta(\omega - \omega_k)$$

$$= \sum_k g_k^2 \exp[i \vec{k} \cdot (\vec{r_1} - \vec{r_2})] \delta(\omega - \omega_k).$$

$J_{11}(\omega)$ and $J_{12}(\omega)$ come from located and collective interaction with the environment, respectively. We consider in detail the particular case of ohmic bath, which is important for charged interstitial in metals, heavy particle in normally conducting and superconductive environment [16]. In addition, ohmic bath displays the richest variety of behavior as a function of the parameters [17]. We specialize to linear, isotropic dispersion relation: $\omega_q = sq$, where $s$ is the particle velocity of the boson environment. The wave function distribution in the qubits are assumed $\sim \exp[-r^2/(2l^2)]$, $l$ is the qubit size [18–20]. We get,

$$J_{11}(\omega) = \int \frac{dk}{2} |g_k(r_1)|^2 \delta(\omega - \omega_k) dk = 2\alpha \omega \theta(\omega_c - \omega),$$

$$J_{12}(\omega) = \int \frac{dk}{2} |g_k(r_1)|^2 \cos[\vec{k} \cdot (\vec{r_1} - \vec{r_2})] \delta(\omega - \omega_k) dk$$

$$= 2\alpha \omega \sin(\omega / \omega_d) / (\omega / \omega_d) \theta(\omega_c - \omega),$$

where $\alpha$ is the dimensionless coupling constant, $\omega_c = s / l$ and $\omega_d = s / d$ ($l$ is the qubit size and $d = |r_1 - r_2|$ is the center-to-center distance between two qubits) and $\theta(x)$ is the usual step function.

Taking into account of the correlation between qubits and environment, we apply a canonical transformation, $H' = \exp(T) H \exp(-T)$ with the generator [21]:

$$T = \sum_{j,k} \frac{g_k(r_j)b_k^+ - g_k(r_j)b_k}{2\omega_k} \xi_{jk} \sigma_{jx}.$$ 

Then decompose the transformed Hamiltonian $H'$ into three parts:

$$H' = H'_0 + H'_1 + H'_2,$$

$$H'_0 = -\frac{1}{2} \sum_j \eta_j \Delta_j \sigma_{jz} + \sum_k \omega_k b_k^+ b_k$$

$$- \sum_{j,k} [g_k(r_j)]^2 \xi_{jk} (2 - \xi_{jk}),$$

$$H'_1 = \sum_{j,k} \eta_j \Delta_j \frac{\xi_{jk}}{\omega_k} \left[ g_k^2(r_j) b_k^+ \sigma_{jz} + g_k(r_j)b_k \sigma_{j+} + \lambda \sigma_{1z} \sigma_{2z} \right],$$

$$H'_2 = -\frac{1}{2} \sum_j \Delta_f \sigma_{jz}$$

$$\times \left\{ \cosh \left[ \sum_k \xi_{jk} (g_k^2(r_j) b_k^+ - g_k(r_j)b_k) \right] - \eta_j \right\}$$

$$- \frac{1}{2} \sum_j i \Delta_f \sigma_{jy} \sinh \left[ \sum_k \xi_{jk} (g_k^2(r_j) b_k^+ - g_k(r_j)b_k) \right]$$

$$+ \frac{1}{2} \sum_{j,k} i \Delta_f \sigma_{jy} \eta_j \xi_{jk} (g_k^2(r_j) b_k^+ - g_k(r_j)b_k),$$

with

$$\eta_j = \exp \left[ -\frac{1}{2} \sum_k \left( \frac{|g_k| \xi_{jk}}{\omega_k} \right)^2 \right],$$

$$\xi_{jk} = \frac{\omega_k}{\omega_k + \eta_j \Delta_j}.$$
Here $\sigma_{jk} = (\sigma_{jx} \pm i \sigma_{jy})/2$, $H_0'$ is the diagonal Hamiltonian of the qubits and the environment, $H_1'$ and $H_2'$ are the Hamiltonian of the qubit-environment interaction. The purpose of our unitary transformation is to find a better way to divide the transformed Hamiltonian into unperturbed part $H_0'$, which can be treated exactly, and $H_1' + H_2'$, which may be treated by perturbation theory. By choosing the form of $\xi$ (Eq. (14)) and $\eta$ (Eq. (15)), it is possible to treat $H_1'$ and $H_2'$ as perturbation because of the following reasons: 1) if we treat the coupling term in the original Hamiltonian $H$ as the perturbation, the dimensionless expanding parameter is $\sum_k g_k^2/\omega_k^2$. For Ohmic bath, it is $2\alpha \int d\omega/\omega$ which is logarithmic divergent in the infrared limit, but for the coupling term in transformed Hamiltonian $H'$, the expanding parameter is $\sum_k g_k^2/\omega_k^2 \sim 2\alpha \int d\omega/\omega + \eta \Delta^2$, which is finite in the infrared limit; 2) $H_1'$ can be treated as perturbation because it satisfied $H_1' \{g \} = 0$. It is shown that the ground state energy correction of $H_1'$ is zero. $H_2'$ may be omitted because its contribution is zero at second order of $g_k$. Experiments are usually performed at low temperature below $20$ mK. At such a low temperature, the multi-excitation process of bosonic environment for the second and higher order of $g_k$ is weak enough to be negligible. Ignoring the multi-boson assisted interaction, we can rewrite the total Hamiltonian as $H' = H_0' + H_1'$. Comparing $H_1$ to $H_1'$, the term in $H_1$ is replaced by the similar rotating-wave approximation term in $H_1'$. It is important for later convolution operation in master equation. The qubit-environment coupling strength $g_k(r_j)$ in $H_1$ is replaced by $\eta \Delta_j \xi_{jk} g_k(r_j)/\omega_k$ in $H_1'$. Since $\eta \Delta_j/\omega_k < 1$, it seems that the counter rotating-wave terms diminish the energy exchange between the qubits and environment. The last term $\lambda \sigma_3^2 \sigma_1^2$ of $H_1'$ describes the induced interaction between two qubits by common environment. The strength $\lambda$ is of the order of $|g_k|^2$.

In the interaction picture,

$$V'_j(t) = V_j^0(t) + V_j^\lambda(t), \quad (18)$$

with

$$V_j^0(t) = \sum_{jk} \eta \Delta_j \xi_{jk} g_k^\dagger(r_j) \sigma_j^- \exp \left[ i(\omega_k - \eta \Delta_j) t \right]$$

+ $\sum_{jk} \eta \Delta_j \xi_{jk} g_k(r_j) \sigma_j^+ \exp \left[ -i(\omega_k - \eta \Delta_j) t \right], \quad (19)$$

$$V_j^\lambda(t) = \lambda \{ \sigma_{1+} \exp \left[ i \eta \Delta_j t \right] + \sigma_{1-} \exp \left[ -i \eta \Delta_j t \right] \} \times \{ \sigma_{2+} \exp \left[ i \eta \Delta_j t \right] + \sigma_{2-} \exp \left[ -i \eta \Delta_j t \right] \}. \quad (20)$$

We consider in general that a system denoted by $S$ interacts with an environment or bath denoted by $B$. The combined density operator is denoted as $\rho_{SB}$. The reduced density operator for the system $S$ is obtained by taking a trace over the environment coordinates, i.e.,

$$\rho_S = Tr_B(\rho_{SB}). \quad (21)$$

The equation of motion for $\rho_{SB}'(t)$ is given by

$$\frac{d}{dt} \rho_{SB}'(t) = -i[\rho_{SB}'(t), \rho_{SB}'(t)]. \quad (22)$$

After $T$ transformation,

$$\frac{d}{dt} \rho_{SB}'(t) = -i[V_j(t), \rho_{SB}'(t)]. \quad (23)$$

This equation can be formally integrated, and we obtain

$$\rho_{SB}'(t) = \rho_{SB}'(t_0) - i \int_{t_0}^{t} [V_j(t'), \rho_{SB}'(t')] dt'. \quad (24)$$

Here $t_0$ is an initial time for the interaction starts and we suppose $t_0 = 0$. On substituting $\rho_{SB}'(t)$ into equation (23), we find the equation of motion

$$\frac{d}{dt} \rho_{SB}'(t) = -i[V_j(t), \rho_{SB}'(t)] - \int_{0}^{t} [V_j(t'), [V_j(t'), \rho_{SB}'(t')]] dt'. \quad (25)$$

Look for a solution of above equation of the form $\rho_{SB}'(t) = \rho_{S}(t)\rho_{B}(0) + \rho_c(t)$, where $\rho_c(t)$ is the higher order term in $g_k$ and $Tr_B \left[ \rho_c(t) \right] = 0$ [22]. Substitute for $\rho_{SB}'(t)$ into the integrand of equation (25), and retain terms up to order $g_k^2$, we have

$$\frac{d}{dt} \rho_{S}'(t) = -i[V_j(t), \rho_{S}'(t)]$$

$$- \int_{0}^{t} Tr_B[V_j^0(t'), [V_j^0(t'), \rho_{S}^0(0)]] dt'. \quad (26)$$

For storing entanglement better, we assume two similar qubits and fix $\Delta_1 = \Delta_2 = \Delta$. As a result, $\eta_1 = \eta_2 = \eta$. In the resonating wave approximation between the two qubits, $V_j^\lambda(t)$ can be presented as

$$V_j^\lambda(t) = \lambda \{ \sigma_{1+} \sigma_{2-} + \sigma_{1-} \sigma_{2+} \}. \quad (27)$$

The master equation is obtained as,

$$\frac{d}{dt} \rho_S'(t) = -i\lambda [\sigma_{1+} \sigma_{2-} + \sigma_{1-} \sigma_{2+}], \rho_S'(t)$$

$$- \int_{0}^{t} Tr_B[V_j^0(t'), [V_j^0(t'), \rho_{S}^0(0)]] dt'. \quad (28)$$
\[
\frac{d}{dt} \rho_S(t) = -i\lambda [\sigma_1 + \sigma_2] + [\sigma_1 - [\sigma_2], \rho_S(t)] + \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 n_k \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \frac{d'}{dt'} \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
- \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 n_k \rho_S(t') \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \exp \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
+ \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 n_k \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \frac{d'}{dt'} \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
- \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 (n_k + 1) \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \frac{d'}{dt'} \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
+ \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 (n_k + 1) \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \exp \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
+ \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 (n_k + 1) \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \rho_S(t') \frac{d'}{dt'} \left[ i(\eta \Delta - \omega_k)(t - t') \right] dt' \\
- \int_0^t \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 (n_k + 1) \rho_S(t') \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \exp \left[ -i(\eta \Delta - \omega_k)(t - t') \right] dt'. \tag{31}
\]

\[P \rho_S^{-1}(P) - \rho_S(0) = -i\lambda [\sigma_1 + \sigma_2 - \sigma_1 - \sigma_2], \rho_S^{-1}(P) \rightleftharpoons \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] \frac{\rho_S^{-1}(P)}{P - i(\eta \Delta - \omega_k)} \\
+ \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 \left[ \sum_j g_k(r_j) \sigma_{j-} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j+} \right] \frac{\rho_S^{-1}(P)}{P - i(\eta \Delta - \omega_k)} \\
\times \frac{\rho_S^{-1}(P)}{P + i(\eta \Delta - \omega_k)} \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right] - \sum_k \left( \eta \frac{\Delta \xi_k}{\omega_k} \right)^2 \frac{\rho_S^{-1}(P)}{P + i(\eta \Delta - \omega_k)} \left[ \sum_j g_k(r_j) \sigma_{j+} \right. \\
\left. \sum_j g_k(r_j) \sigma_{j-} \right]. \tag{32}\]

Substituting \(V(t)\) into equation (28) and supposing the environment modes in thermalization, \(T_{RB}\) are given by:

\[T_{RB}[b_k^+ b_k \rho_B] = T_{RB}[b_k \rho_B b_k^+] = n_k, \tag{29}\]

\[T_{RB}[b_k^+ b_k^+ \rho_B] = T_{RB}[b_k^+ \rho_B b_k^+] = n_k + 1. \tag{30}\]

We get,

\[\text{see equation (31) above.}\]

In this equation, the \(n_k\) and \(n_k + 1\) terms on the right hand side describe the average thermal excitation of the environment, with the rate \(n_k\) which depends on the temperature. In this work, we consider the limit of zero temperature, \(n_k = 0\).

The matrix equation is solved in the representation spanned by the basis of the standard two-qubit product states \(|1\rangle = |1\rangle_{1z}|1\rangle_{2z}, |2\rangle = |1\rangle_{1z}|2\rangle_{2z}, |3\rangle = |1\rangle_{1z}|1\rangle_{2z}, |4\rangle = |1\rangle_{1z}|2\rangle_{2z}, |5\rangle = |1\rangle_{1z}|1\rangle_{2z}.\) These parameters contribute not only to the Lamb shift of the qubit levels, but also to the decay rate. The parameters \(\Gamma_{ij}^\pm\) arise from the induced interaction between the qubits through the common environment. The decay rates are dependent on the processes, as seen from \(\Gamma_{ij}^\pm\) and \(\Gamma_{ij}^\mp\). According to the Kronecker product property and the technique of Lyapunov matrix equation in matrix theory [24], expand the matrix into vector along rows of the
matrix from two sides of master equation,
\[
\begin{align*}
&\{PI_{16\times 16} + \Gamma_{11} I_{4\times 4} \otimes (\sigma_1 + \sigma_2 - I_{2\times 2})^T \\
+ &\Gamma_{12} I_{4\times 4} \otimes (\sigma_1 + \sigma_2 - I_{2\times 2})^T + \Gamma_{21} I_{4\times 4} \otimes (\sigma_1 - \sigma_2 + I_{2\times 2})^T \\
+ &\Gamma_{11}^\dagger (\sigma_1 + \sigma_2 - I_{2\times 2})^T + \Gamma_{22} I_{4\times 4} \otimes (\sigma_1 - \sigma_2 + I_{2\times 2})^T \\
+ &\Gamma_{11}^\dagger (\sigma_1 + \sigma_2 - I_{2\times 2})^T + (\Gamma_{11}^\dagger + \Gamma_{12})[\sigma_1 - I_{2\times 2} \otimes (\sigma_1 - \sigma_2 + I_{2\times 2})^T] \\
+ &\Gamma_{12}^\dagger (\sigma_1 + \sigma_2 - I_{2\times 2} \otimes (\sigma_1 + \sigma_2 + I_{2\times 2})^T)] \\
\} &= \rho_{\text{S}}(0).
\end{align*}
\]

The 4 × 4 matrix equation transformed into 16 × 16 vector equation with the form
\[
U(P)_{16\times 16} V\text{ec}[\rho_{\text{S}}'(P)] = V\text{ec}[\rho_{\text{S}}'(0)]
\]
where \(V\text{ec}[\rho_{\text{S}}'(P)]\) is the vector of row expanding of matrix \(\rho_{\text{S}}'(P)\). The solution formally is
\[
V\text{ec}[\rho_{\text{S}}'(P)] = U(P)^{-1}_{16\times 16} V\text{ec}[\rho_{\text{S}}'(0)].
\]

Inverse Laplace transformation to time parameter space,
\[
\mathcal{L}^{-1}V\text{ec}[\rho_{\text{S}}'(P)] = \mathcal{L}^{-1}U(P)^{-1}_{16\times 16} V\text{ec}[\rho_{\text{S}}'(0)],
\]
i.e.,
\[
V\text{ec}[\rho_{\text{S}}'(t)] = \mathcal{L}^{-1}U(P)^{-1}_{16\times 16} V\text{ec}[\rho_{\text{S}}'(0)].
\]

\(\mathcal{L}^{-1}U(P)^{-1}_{16\times 16}\) can be obtained (see Appendix A), and the master equation is solved. In the Markovian approximation [15,24], the integral over the time delay \(t\) in equation (31) contains function from environment which decay to zero over a short correlation time \(\tau_c\). Over this short time-scale the density operator would hardly have changed from \(\rho(t)\), thus \(\rho(t-t)\) is replaced by \(\rho(t)\) and extend the time integral to infinity. So the integration over \(t\) be performed and is obtained,
\[
\lim_{t^{-\infty}} \int dt' \rho(t-t')e^{i\omega t} \approx \rho(t) \left[ \pi \delta(x) + i \frac{\varphi}{x} \right]
\]
where \(\varphi\) stands for Cauchy principal value. The spontaneous damping rates of the system possess the form such as \(\gamma_{ij} = \sum_k |g_k(r_i)|^2 \delta(k - k_j)\), which is a constant with reference to Rabi oscillation frequency of the \(j\)th bare qubit. Compared with Markovian approximation, decoherence rates \(\gamma_{\omega}\) in our results are dependent on frequency, which record the effect of the environment or Lamb shift. And the decoherence rates for a variety of processes are different, some become small, some become large. Virtually, the decoherence rates and the Lamb shifts of the two subsystems are both determined by the renormalization of the terms \(\Gamma_{jj}^\pm = \sum_k \frac{(\eta\Delta k)^2 |g_k(r_j)|^2}{\rho_{\text{S}}(\eta\omega - \eta |\Delta k|)}\)

\((j = 1, 2)\) and \(\Gamma_{ij}^\pm = \sum_k \frac{(\eta\Delta k)^2 |g_k(r_j)|g_k(r_i)|^2}{\rho_{\text{S}}(\eta\omega - \eta |\Delta k|)}\). That is more general and physical.

Therefore, the reduced density matrix \(\rho_{\text{S}}'(t)\) in the Schrödinger picture is obtained \(\rho_{\text{S}}'(t) = \exp(-iH_0t)\rho_{\text{S}}'(P)\exp(iH_0t)\), the matrix form is
\[
\rho_{\text{S}}'(t) = \left[ \begin{array}{cc}
\exp \left(\frac{i\eta t}{2}\right) & 0 \\
0 & \exp \left(-i\frac{i\eta t}{2}\right)
\end{array} \right] \otimes \left[ \begin{array}{cc}
\exp \left(\frac{-i\eta t}{2}\right) & 0 \\
0 & \exp \left(i\frac{-i\eta t}{2}\right)
\end{array} \right] \rho_{\text{S}}'(P) \left[ \begin{array}{cc}
\exp \left(-i\frac{\eta t}{2}\right) & 0 \\
0 & \exp \left(i\frac{\eta t}{2}\right)
\end{array} \right].
\]

Transform \(\rho_{\text{S}}'(t)\) into \(\rho_{\text{S}}(t)\) through
\[
\rho_{\text{S}}(t) = Tr_B[\exp(-S)\rho_{\text{S}}'(t)\rho_B(0)\exp(S)],
\]
denoting \(X_1 = \sum_k \frac{\xi_k}{\omega_k} |g_k(r_1)|b_k^+ - g_k(r_1)bk\), \(X_2 = \sum_k \frac{\xi_k}{\omega_k} |g_k(r_2)|b_k^+ - g_k(r_2)bk\), so
\[
\rho_{\text{S}}(t) = Tr_B [(\cosh X_1 - \sinh X_1 \sigma_{1x}) \otimes (\cosh X_2 - \sinh X_2 \sigma_{2x}) \rho_{\text{S}}'(t) \rho_B(0)\rho_{\text{S}}(t) \rho_B(0) \rho_{\text{B}}(0)],
\]
\[
\quad \times (\cosh X_2 + \sinh X_2 \sigma_{2x}) \rho_{\text{S}}'(t),
\]
\[
= \left[ \begin{array}{cc}
1 + \frac{\eta^2}{2} & \eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2} \\
\frac{1}{2} & \frac{\eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2}}{8}
\end{array} \right] \rho_{\text{S}}'(t),
\]
\[
+ \left[ \begin{array}{cc}
1 + \frac{\eta^2}{2} & \eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2} \\
\frac{1}{2} & \frac{\eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2}}{8}
\end{array} \right] \rho_{\text{S}}'(t)
\]
\[
\times \left( I_{2\times 2} \otimes \sigma_{2x} \right) \rho_{\text{S}}'(t),
\]
\[
+ \left[ \begin{array}{cc}
1 + \frac{\eta^2}{2} & \eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2} \\
\frac{1}{2} & \frac{\eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2}}{8}
\end{array} \right] \rho_{\text{S}}'(t)
\]
\[
\times \left( I_{2\times 2} \otimes \sigma_{1x} \right) \rho_{\text{S}}'(t)
\]
\[
+ \left[ \begin{array}{cc}
1 + \frac{\eta^2}{2} & \eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2} \\
\frac{1}{2} & \frac{\eta^2 (\eta_{12})^2 + \eta^2 (\eta_{12})^{-2}}{8}
\end{array} \right] \rho_{\text{S}}'(t), \sigma_{1x} \otimes \sigma_{2x}, \rho_{\text{S}}'(t), \sigma_{1x} \otimes \sigma_{2x}.
\]

Until now, we have not fixed what the initial state of the two qubits.

3 The result and discussion

We assume that at \(t = 0\), the two qubits and environment are described by the state \(\rho_{\text{S}}(0) = \exp(-T)\rho_{\text{S}}(0)\otimes \rho_{\text{B}}(0)\).
|0⟩⟨0| \exp(T), where |0⟩ is the vacuum state of the environment. Use Wootters’s concurrence to quantify the degree of entanglement [25,26]. Let \( \rho \) be density matrix of the pair of qubits in the standard basis. The concurrence may be calculated explicitly from the density matrix \( \rho : C = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \), where the quantities \( \lambda_i \) are the eigenvalues of the matrix \( M = \rho (\sigma^x \otimes \sigma^y) \rho^* (\sigma^x \otimes \sigma^y) \), arranged in decreasing order. Here \( \rho^* \) denotes the complex conjugation of \( \rho \).

In the following calculations, we choose the qubit size \( l \) as 100 nm (approximate size for the dot in Ref. [27]), \( \omega_c \approx 0.05(\text{ps})^{-1} \), corresponding to an energy of \( 32.5 \mu\text{eV} \). Times in units of \( \omega_c^{-1} \). Fix the energy splitting of each qubit, \( \Delta = 0.2\omega_c = 6.5 \mu\text{eV} \).

The Bell states in the quantum teleportation are quite important, such as in EPR coincidence experiments [28]. So we choose four Bell states as the initial states in this work. Define the four Bell states as \(|\Phi_{12}^\pm\rangle = (|1\rangle_1|2\rangle_2 \pm |2\rangle_1|1\rangle_2)/\sqrt{2} \) and \(|\Psi_{12}^\pm\rangle = (|1\rangle_1|1\rangle_2 \pm |1\rangle_1|1\rangle_2)/\sqrt{2} \). The density matrix operates are given by \( \rho_{1,2} = |\Phi_{12}^\pm\rangle \langle \Phi_{12}^\pm| \) and \( \rho_{3,4} = |\Psi_{12}^\pm\rangle \langle \Psi_{12}^\pm| \). That is easily expressed in the following form,

\[
\rho_{1,2} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & \pm 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pm 1 & 0 & 0 & 1
\end{pmatrix}, \quad \rho_{3,4} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & \pm 1 \\
0 & \pm 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (42)
\]

However, four Bell states are the eigenstates of the operator \( V^\lambda = \lambda (\sigma_1^x \sigma_2^- + \sigma_1^- \sigma_2^+) \), the effect of \( V^\lambda \) on the concurrence can not exhibited in this work.

To start with, we consider the time evolution of concurrence for fixed distance and weak dissipative environment \( \alpha = 0.01 \), which is corresponding to the atom in the electromagnetic field \( \alpha = 1/137 \), in Figure 1a. For the initial states \(|\Phi_{12}^\pm\rangle \), two curves of concurrence coincide as \( d \approx 2000 \text{ nm} \). To further increase the distance, there will be no any change in the curves. While for the initial states \(|\Psi_{12}^\pm\rangle \), two curves are almost overlapped until \( d \approx 30000 \text{ nm} \). Under the influence of weak dissipative environment, the decay of concurrence for the initial states \(|\Phi_{12}^\pm\rangle \) is accelerated by a factor of almost two as compared to \(|\Psi_{12}^\pm\rangle \) for enough separation. The same result is obtained by Mintert [8], in which they assume that two subsystems are sufficiently separated from each other, therefore each subsystem interacts independently with the environment. So the model that each subsystem interacts independently with its environments is a well justified assumption when the particles composing the system are far enough apart.

The parameters \( I_{ij}^\pm \) depend on the interqubit separation and determine the collective effect of the system. In order to see the influence of interqubit distance on the entanglement, we plot concurrence of \( \omega_c t = 30 \) in weak dissipative environment \( \alpha = 0.01 \) for four Bell states as a function of the distance from 100 nm to 8000 nm in Figure 1b. It shows that the effects of distance on the concurrence are different for the different initial states. In Bell states \(|\Phi_{12}^\pm\rangle \), the concurrences firstly decrease in a small range of distance 100–1000 nm then stabilize for increasing distance. While for Bell states \(|\Psi_{12}^\pm\rangle \), the concurrences present damping oscillation, overlapped for enough large separation. The peak and valley of oscillation are reversed between the two states. That is to say, the destruction and instruction of entanglement oscillates with the distance in the initial states \(|\Psi_{12}^\pm\rangle \). For sufficient large separation \( (d \gg l) \), the effects of distance on the concurrence get small, i.e. the parameters \( I_{ij}^\pm \) become very small \((I_{ij}^\pm \approx 0) \). The parameters \( I_{ij}^\pm \) indicate that the spontaneous emission from one of the two qubits influences the spontaneous emission of the other. Owing to the induced coupling, the population is coherently transferred back and forth from one qubit to the other one through the dissipative environment, just as shown from the oscillation of the concurrence as distance for the state \(|\Psi_{12}^\pm\rangle \). While for the state \(|\Phi_{12}^\pm\rangle \), the entanglement decays due to...
the decoherence of double excited component. The result illuminates from another aspect that when the particles composing the system are far enough apart, it can be consider as each subsystem interacts independently with its environment.

The interaction constant of atom with a continuum of electromagnetic field modes surrounding approximately is fine structure constant $\alpha = 1/137$. However, the electron-phonon coupling constant $\alpha = 0.02-0.07$ was used to explain the inelastic current in GaAs/AlGaAs heterostructure double QD samples [27,29]. So we choose $\alpha = 0.01$ in figure (a) for weak coupled case, in which the Markovian approximation works well, and $\alpha = 0.05$ in figure (b) for the strong coupled case to explain the non-Markovian dynamics. Figures 2–5 show the time evolution of the concurrence for four Bell states. In Figures 2a and 2b, we plot the dynamics of concurrence for the maximally entangled state $|\Phi^+_{12}\rangle$ with distance from 100 nm to 2000 nm. From Figure 2a, when the distance $d = 100$ nm, the concurrence is reduced with a slight oscillation. As the interqubit distance becomes large progressively, the oscillation amplitude decreases gradually, eventually disappears and the exponential decay appears. The concurrence shown in Figure 2b decays faster in all range of distance than the case of $\alpha = 0.01$. The most interesting consequence of the collective decoherece is the entanglement revival for interqubit distance from 100 nm to 500 nm. The concurrence damps with oscillation at first, with time increasing, it gets to zero, “death of the entanglement”, and remains disentangled for a short while, somewhat counterintuitively, the entanglement revives to evolve further. But every revival
amplitude is smaller than its previous one’s. The entanglement may revives only when two qubits are close. If the separation large enough, the entanglement completely vanishes in finite time (sudden death) without any revivals. Similar phenomenon is observed by Ficek et al. [30] The time evolution of the concurrence in the initial state $|\Psi_{12}\rangle$ is shown in Figure 3, the properties of which are almost the same as that of Figure 2.

The time evolution of the concurrence for the initially state $|\Psi_{12}\rangle$ with distance from 100 nm to 8000 nm are shown in Figures 4a and 4b, corresponding to coupling strength $\alpha = 0.01$ and 0.05 respectively. We find that these curves of concurrence from $d = 100$ nm to $d = 8000$ nm are exponential decay in Figure 4a, i.e. the disentanglement process lasts for an infinite time period in the weak dissipative environment. However, the concurrence always presents the sudden-death of entanglement when $\alpha = 0.05$ in Figure 4b. We also find that concurrence oscillates with distance, so the separation does not always destroy the quantum entanglement in the initial state $|\Psi_{12}\rangle$. When we use the formal solution $\rho_{S}^{I}(t)$ in the initial state $|\Psi_{12}\rangle$ to discuss the disentanglement dynamics of the two qubit system. $\rho_{S}^{I}(t)$ can be expressed simply as,
Bell states | \begin{array}{ccc}
1 - \gamma^{-1} & 0 & 0 \\
0 & 1 - \gamma^{-1} & 0 \\
0 & 0 & 1 - \gamma^{-1}
\end{array}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\end{array}
\right)
\rho'_{s}(t) = \left( \begin{array}{ccc}
1 - \gamma^{-1} & 0 & 0 \\
0 & 1 - \gamma^{-1} & 0 \\
0 & 0 & 1 - \gamma^{-1}
\end{array}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\frac{1}{P + G_{11} + G_{11} + G_{12} + G_{12}}
\end{array}
\right)
\right)
\end{equation}

From the derivation, the decay rate for |Ψ_{12}^+\rangle is denoted as 2[γ_{11}(ω) + γ_{12}(ω)], so the induced interaction between two qubits through the common environment formally increases the decay rate. The initial state |Ψ_{12}^+\rangle is the symmetric state, which is a fast radiative one. So the result corresponds to the superaddition state or bright state in Dicke model [31]. Compared with the corresponding cases in Figure 2, the oscillation of the concurrence as time in the case of short-distance is disappeared.

In Figure 5a, we show that in the initial state |Ψ_{12}^-\rangle for α = 0.01, the entanglement evolve with exponential decay for all distance from d = 100 nm to d = 8000 nm. In contrast with Figure 4a, when the distance between two qubits change from 100 nm to 2000 nm, the decay rate of the concurrence cuts down gradually. Some papers drawn the conclusion that the entanglement can be sustained much longer when two subsystem are coupled to a common bath than to individually independent baths [13,14]. In this paper, we think two subsystem coupled to individually independent baths corresponds to a extreme case of two subsystem coupled to a common bath, that is the distance between two qubits infinite. From our results, the above conclusion apply only in some specific cases, such as initial state |Φ_{12}^\pm\rangle. The sudden death of entanglement appears for α = 0.05 in Figure 5b. We also obtain ρ'_{s}(t) simply,

\begin{equation}
\text{see equation (43) above.}
\end{equation}

In comparison with |Ψ_{12}^+\rangle, the decay rate for |Ψ_{12}^-\rangle is 2[γ_{11}(ω) - γ_{12}(ω)]. In the Dicke model [31,32], the antisymmetric atomic state |Ψ_{12}^-\rangle is a decoherence-free or nonradioactive one, just as the case of γ_{12}(ω) = γ_{12}(ω). Our result indicates that the initial condition of antisymmetric atomic state |Ψ_{12}^-\rangle inhibit formally the spontaneous emission. If the two qubits are separated large enough, γ_{12}(ω) ≈ 0, the evolution of the concurrence for the two Bell states |Ψ_{12}^\pm\rangle are overlapped.

When the interqubit distance is sufficiently large, comparing Figures 2–5, in the weak dissipative environment α = 0.01 the concurrence for Bell states |Φ_{12}^\pm\rangle decays faster than Bell states |Ψ_{12}^\pm\rangle. While in the strong dissipative environment α = 0.05, it can seen that the entanglement sudden death always happens, no matter which entangled state that the qubits are initially in. We see that the sudden-death time of entanglement in |Φ_{12}^\pm\rangle are always smaller than in |Ψ_{12}^\pm\rangle in the same condition. In other words, the entanglement can be sustained much longer when two subsystems are initialized in Bell states |Ψ_{12}^\pm\rangle than in Bell states |Φ_{12}^\pm\rangle. The Bell states |Φ_{12}^\pm\rangle are composed by the double excited state [|1\rangle|1\rangle]22, and the ground state [|0\rangle|0\rangle]22. While the entangled states |Ψ_{12}^\pm\rangle allows either qubit 1 or qubit 2 to be excited with equal probabilities. In this sense, the double excitation component in the initial state makes entanglement more sensitive to the quantum fluctuation of the environment. We also found that the entanglement revival don’t exist in a case of initially only one qubit excited [33].

4 Conclusion
We investigate the non-Markovian disentanglement dynamics of two qubits coupled to a common bosonic environment by the means of perturbation based on unitary transformation. It is discussed that the influence of coupling strength with the environment on the disentanglement dynamics for different initial states (four Bell states) by varying interqubit distance. As a consequence, we find that, for Bell state |Ψ_{12}^\pm\rangle, the entanglement measured by concurrence display exponential decay for all range of distance in the case of weak coupling with environment, but in the case of strong coupling, the entanglement sudden-death always happens. For Bell state |Φ_{12}^\pm\rangle, in weak dissipative environment the concurrence firstly decay with oscillation. To further increase distance, it exhibits exponential decay. While in strong dissipative environment, as increasing the interqubit distance concurrence shows entanglement revival firstly, then shows sudden death. The life of entanglement in our result is obtained from the
\[ \mathcal{L}^{-1} \frac{1}{P + \Gamma_{11} + \Gamma_{12}} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(i\omega t + 0^+) \exp(i\omega t + 0^+) \] 
\[ \times \frac{1}{\omega - R_{11}(\omega + \eta \Delta) - R_{12}(\omega + \eta \Delta) - i\gamma_{11}(\omega + \eta \Delta) - i\gamma_{12}(\omega + \eta \Delta)} d\omega \] 
\[ = \exp[\omega_1 t - \gamma_{11}(\omega_1 + \eta \Delta)t - \gamma_{12}(\omega_1 + \eta \Delta)t] \] 
\[ (A.9) \]
\[ (A.10) \]

physical parameter: coupling constant \( \alpha \), energy splitting \( \Delta \), interqubit distance \( d \) and cut-off frequency \( \omega_c \). Finally, we hope that this work will stimulate more experimental and theoretical works to detect and investigate the relation of entanglement and interqubit distance in quantum information and quantum computation.

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Appendix A

In this appendix, we give details of how to do the inversion of Laplace transformation. \( U(P)^{-1} \) is composed of the matrix element: \( 1/P, 1/(P + \Gamma_{11} + \Gamma_{12}), 1/(P + 2\Gamma_{11}), 1/(P + \Gamma_{22} + \Gamma_{12}), 1/(P + \Gamma_{22} + \Gamma_{21}), 1/(P + \Gamma_{22} + \Gamma_{12}) \) etc., with \( \Gamma_{11} = \sum_k ((\eta \Delta \xi_k^2)^2|g_k(r_1)|^2)/(P + i(\omega_k - \eta \Delta)), \Gamma_{12} = \sum_k ((\eta \Delta \xi_k^2)^2|g_k(r_1)|^2)/(P + i(\omega_k - \eta \Delta)), (i, j = 1, 2) \). Then \( \mathcal{L}^{-1}U(P)^{-1} \) is the inversion of every matrix element. As we know, \( \mathcal{L}^{-1} \frac{1}{P + \Gamma_{11} + \Gamma_{12}} \) etc. through the following method.

\[ \mathcal{L}^{-1} \frac{1}{P + \Gamma_{11} + \Gamma_{12}} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(Pt) \] 
\[ \times \frac{1}{\sigma - \infty} P + \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \] 
\[ + \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \] 
\[ \times \frac{1}{P - i(\omega_k - \eta \Delta)} \] 
\[ (A.1) \]

Then changing \( P \) to \( i\omega + 0^+ \) [34],

\[ \mathcal{L}^{-1} \frac{1}{P + \Gamma_{11} + \Gamma_{12}} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(Pt) \] 
\[ \times \frac{1}{\sigma - \infty} P + \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \] 
\[ + \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \] 
\[ \times \frac{1}{P - i(\omega_k - \eta \Delta)} \] 
\[ (A.2) \]

and imaginary parts of \( \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \),

\[ R_{ii}(E) = \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \] 
\[ = (\eta \Delta)^2 \int_{0}^{\infty} d\omega' \frac{J_{ii}(\omega')}{(E - \omega')(\omega' + \eta \Delta)^2}, \] 
\[ (A.5) \]
\[ \gamma_{ii}(E) = \pi \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 \delta(E - \omega_k) \] 
\[ = \pi \left( \frac{\eta \Delta}{E + \eta \Delta} \right)^2 J_{ii}(E), \] 
\[ (A.6) \]

and

\[ R_{ij}(E) = \sum_k (\eta \Delta \xi_k^2)^2|g_k(r_1)|^2 g_k(r_j) \] 
\[ = (\eta \Delta)^2 \int_{0}^{\infty} d\omega' \frac{J_{ij}(\omega')}{(E - \omega')(\omega' + \eta \Delta)^2}, \] 
\[ (A.7) \]
\[ \gamma_{ij}(E) = \pi \sum_k (\eta \Delta \xi_k^2)^2 g_k(r_1)g_k(r_j) \delta(E - \omega_k) \] 
\[ = \pi \left( \frac{\eta \Delta}{E + \eta \Delta} \right)^2 J_{ij}(E), \] 
\[ (A.8) \]

where \( \varphi \) stands for Cauchy principal value.

see equations (A.9), (A.10) above

where \( \omega_1 \) is the solution of equation \( \omega - R_{11}(\omega + \eta \Delta) - R_{12}(\omega + \eta \Delta) = 0 \) and is the Lamb shift of the qubit 1 due to the interaction with the common environment.

The Laplace inversion terms of other matrix elements may be obtained in the same way.

References

34. G.D. Mahan, Many-Particle physics (World Scientific, New York, 1990)