Non-Markovian dynamics of a dissipative two-level system: Nonzero bias and sub-Ohmic bath

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The quantum dynamics of the dissipative two-level system with nonzero bias and sub-Ohmic bath is studied by means of the perturbation approach based on a unitary transformation. It has been shown that for the sub-Ohmic bath it is necessary to use the non-Markovian approach, especially for the short time behavior of the coupled system and environment. The nonequilibrium correlation $P(t)$ has been calculated to show that a finite bias may favor the short time coherence. The spectrum of the susceptibility $\chi''(\omega)$ of the sub-Ohmic case may have a double peak structure in the range of $\omega > 0$ when the coupling α is relatively strong. Besides, the coherence-decoherence transition point α_c is determined for different $0 < s \le 1$ by the condition of $\chi''(\omega=0) = \infty$ when $\alpha = \alpha_c$. Finally, we show that Shiba's relation is exactly satisfied in our results.

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I. INTRODUCTION

The physics of a quantum two-level system coupled to dissipative bosonic environment [spin-boson model (SBM)] has attracted considerable attention in last years because it provides a universal model for numerous physical and chemical processes [1,2]. The Hamiltonian of SBM reads (throughout this paper we set $\hbar = 1$)

$$H = -\frac{1}{2}\Delta\sigma_x + \frac{1}{2}\epsilon\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \frac{1}{2}\sum_k g_k(b_k^\dagger + b_k)\sigma_z, \tag{1}$$

where $b_k^\dagger$ (b_k) is the creation (annihilation) operator of boson mode with frequency ω_k and σ_x and σ_z are Pauli matrices to describe the two-level system. ϵ is the bias, Δ is the bare tunneling, and g_k is the coupling between spin and environment.

The essential physics contained in SBM is the competition between the coherent quantum dynamics of the two-level system [the Rabi oscillation described by the first two term of Eq. (1)] and the dissipative effect of the environment which tends to make the dynamics decoherent. The main theoretical interest is to understand how the environment influences the dynamics of the two-level system and, in particular, how dissipation destroys quantum coherence [1–4]. Both the nonequilibrium and equilibrium dynamics are of interest for the different experimental realizations of two-level systems. When the system can be prepared in one of the two states by applying a strong bias for times $t < 0$ and then let it evolve for $t > 0$ in a finite bias $\epsilon \neq 0$, the nonequilibrium correlation function $P(t)$ is of primary interest [1,5]. When the initial state preparation is not realizable, the interest then lies in the susceptibility $\chi(\omega)$ [2,3]. Moreover, the real and imaginary parts of $\chi(\omega)$ should satisfy Shiba's relation [3,6–8].

The effect of the bosonic environment is characterized by a spectral density $J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k) = 2\alpha\omega^s\omega_c^{1-s}\theta(\omega_c - \omega)$ with the dimensionless coupling strength α and the hard upper cutoff ω_c [$\theta(\omega_c - \omega)$ is the usual step function] [9]. The index s accounts for various physical situations: $s=1$ is the Ohmic bath [1,2] but $s<1$ stands for the sub-Ohmic bath [1,2,10–13]. There are suggestions to model a lossy resistor-inductor-capacitor transmission line or the $1/f$ noise found in experiments by the $s<1$ sub-Ohmic bath [14,15].

In this paper, we focus on the sub-Ohmic SBM ($0 < s < 1$) since, in terms of the renormalization group approach, the sub-Ohmic coupling represents a relevant perturbation. In last a few years, the numerical renormalization group method [15–19] and the quantum Monte Carlo method [20] are used for the sub-Ohmic SBM, and their main interest is to study the properties of the delocalized-localized quantum phase transition. But our main interest is different from theirs, that is, our purpose is to understand how the sub-Ohmic bath influences the dynamics of the two-level system and destroys the quantum coherence. Moreover, based on the noninteracting blip approximation [1] there are claims that the two-level system might be always localized in the sub-Ohmic case for zero temperature, thus there should be no coherent dynamics for the sub-Ohmic bath. However, in a previous paper [21] we studied the unbiased ($\epsilon = 0$) sub-Ohmic SBM to show that a finite coherence-decoherence transition point exists for all $0 < s \le 1$. Due to technical difficulties, little result about quantum dynamics of $s<1$ sub-Ohmic bath at zero temperature with finite bias is known.

In this work the analytical approach in Ref. [21] is extended to calculate the non-Markovian dynamics of SBM with sub-Ohmic bath $0 < s \le 1$ and finite bias $\epsilon \neq 0$. Our results will show that for the sub-Ohmic bath a nonzero bias plays an important role in the quantum dynamics and the Markovian approximation is not good, especially for the short time behavior of the coupled system and environment.

II. UNITARY TRANSFORMATION

Here we present a treatment based on the unitary transformation approach. A unitary transformation [22,23] is applied to H, $H' = \exp(S)H\exp(-S)$, and the purpose of the transformation is to take into account the correlation between the spin and bosons, where

$$S = \sum_k \frac{g_k}{2\omega_k}(b_k^\dagger - b_k)[\xi_k\sigma_z + (1-\xi_k)\sigma_0]. \tag{2}$$

Here we introduce in S a constant σ_0 and a k-dependent function ξ_k; their form will be determined later.

The transformation can be done to the end and the result is

$$H' = H_0' + H_1' + H_2', \tag{3}$$

$$H_0' = -\frac{1}{2}\eta\Delta\sigma_x + \frac{1}{2}\epsilon\sigma_z + \sum_k \omega_k b_k^\dagger b_k - \sum_k \frac{g_k^2}{4\omega_k}\xi_k(2-\xi_k) - \sum_k \frac{g_k^2}{4\omega_k}\sigma_0^2(1-\xi_k)^2, \tag{4}$$

$$H_1' = \frac{1}{2}\sum_k g_k(1-\xi_k)(b_k^\dagger + b_k)(\sigma_z - \sigma_0) - \frac{1}{2}\eta\Delta i\sigma_y\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k), \tag{5}$$

$$H_2' = -\sum_k \frac{g_k^2}{2\omega_k}\sigma_0(1-\xi_k)^2(\sigma_z-\sigma_0) - \frac{1}{2}\Delta\sigma_x\left(\cosh\left\{\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right\} - \eta\right) - \frac{1}{2}\Delta i\sigma_y\left(\sinh\left\{\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right\} - \eta\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k)\right), \tag{6}$$

where

$$\eta = \exp\left[-\sum_k \frac{g_k^2}{2\omega_k^2}\xi_k^2\right]. \tag{7}$$

Obviously, H_0' can be solved exactly because the spin and bosons are decoupled. H_0' can be diagonalized by a unitary matrix U,

$$U = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}, \tag{8}$$

$$u = \frac{1}{\sqrt{2}}\left(1 - \frac{\epsilon}{W}\right)^{1/2}, \quad v = \frac{1}{\sqrt{2}}\left(1 + \frac{\epsilon}{W}\right)^{1/2}, \tag{9}$$

where $W = [\epsilon^2 + \eta^2\Delta^2]^{1/2}$.

The diagonalized H_0' is

$$\tilde{H}_0 = U^\dagger H_0' U = -\frac{1}{2}W\sigma_z + \sum_k \omega_k b_k^\dagger b_k - \sum_k \frac{g_k^2}{4\omega_k}\xi_k(2-\xi_k) - \sum_k \frac{g_k^2}{4\omega_k}\sigma_0^2(1-\xi_k)^2. \tag{10}$$

The eigenstate of $\tilde{H}_0$ is a direct product: $|s\rangle|\{n_k\}\rangle$, where $|s\rangle$ is the eigenstate of σ_z, $|s_1\rangle = \binom{1}{0}$ or $|s_2\rangle = \binom{0}{1}$, and $|\{n_k\}\rangle$ is the eigenstate of bosons with n_k phonons for mode k. In particular, $|\{0_k\}\rangle$ is the vacuum state in which $n_k = 0$ for every k. The ground state of $\tilde{H}_0$ is

$$|g_0\rangle = |s_1\rangle|\{0_k\}\rangle, \tag{11}$$

with ground state energy

$$E_g = -\frac{1}{2}W - \sum_k \frac{g_k^2}{4\omega_k}\xi_k(2-\xi_k) - \sum_k \frac{g_k^2}{4\omega_k}\sigma_0^2(1-\xi_k)^2. \tag{12}$$

H_1' is transformed as follows:

$$\tilde{H}_1 = U^\dagger H_1' U = -\frac{1}{2}\sum_k g_k(1-\xi_k)(b_k^\dagger + b_k)\left[\frac{\epsilon}{W}\sigma_z + \sigma_0\right] + \frac{\eta\Delta}{2W}\sigma_x\sum_k g_k(1-\xi_k)(b_k^\dagger + b_k) + \frac{1}{2}\eta\Delta i\sigma_y\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^\dagger - b_k). \tag{13}$$

$\tilde{H}_1$ and $\tilde{H}_2 = U^\dagger H_2' U$ are treated as perturbation and they should be as small as possible. For this purpose σ_0 and ξ_k are determined in such a way

$$\sigma_0 = -\frac{\epsilon}{W}, \tag{14}$$

$$\xi_k = \frac{\omega_k}{\omega_k + W}, \tag{15}$$

which

$$\begin{aligned}\tilde{H}_1 &= \frac{1}{2}\sum_k g_k(1-\xi_k)(b_k^\dagger + b_k)\frac{\epsilon}{W}[1-\sigma_z] + \frac{1}{2}\eta\Delta\sum_k \frac{g_k}{\omega_k}\xi_k[b_k^\dagger(\sigma_x + i\sigma_y) + b_k(\sigma_x - i\sigma_y)] \\ &= \frac{1}{2}(1-\sigma_z)\sum_k Q_k(b_k^\dagger + b_k) + \frac{1}{2}\sum_k V_k[b_k^\dagger(\sigma_x + i\sigma_y) + b_k(\sigma_x - i\sigma_y)],\end{aligned} \tag{16}$$

where $Q_k = \lambda_k\epsilon$, $V_k = \lambda_k\eta\Delta$, and $\lambda_k = g_k/(\omega_k + W)$. Note that $\tilde{H}_1|g_0\rangle = 0$ and this is the key point in our approach. Thus E_g in Eq. (12) is the ground state energy of $\tilde{H}$,

$$E_g = -\frac{1}{2}W - \sum_k \frac{g_k^2}{4\omega_k}\left[1 - \left(\frac{\eta\Delta}{\omega_k + W}\right)^2\right]. \tag{17}$$

The original Hamiltonian can be solved exactly in two limits: one is the weak-coupling limit $\alpha \to 0$ with $E_g(\alpha \to 0) = -\frac{1}{2}\sqrt{\Delta^2 + \epsilon^2}$ and the other is the zero tunneling limit $\Delta \to 0$ with $E_g(\Delta \to 0) = -\frac{1}{2}|\epsilon| - \sum_k g_k^2/4\omega_k$. It is easily to check that E_g in Eq. (17) goes to the correct ground state energy in these two limits.

η in Eq. (7) is the renormalized factor for the tunneling,

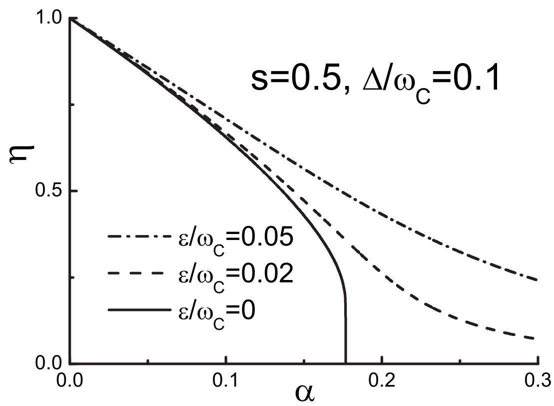

FIG. 1. The renormalized tunneling η vs α relations for different biases: $\epsilon/\omega_c=0$ (solid line), 0.02 (dashed line), and 0.05 (dashed-dotted line).

$$\eta=\exp\left[-\alpha\int_0^1 \frac{x^s dx}{(x+W')^2}\right], \tag{18}$$

where $W'=W/\omega_c$. For some special s values the integration can be done easily, e.g., for $s=1$ $\eta=\exp(-\alpha\{\ln[(1+W')/W']-1/(1+W')\})$ and for $s=1/2$ $\eta=\exp\{-\alpha[\tan^{-1}(1/\sqrt{W'})/\sqrt{W'}-1/(1+W')]\}$. For unbiased case $\epsilon=0$, there is a delocalized-localized quantum phase transition [21]. Figure 1 shows that this transition point for $s=1/2$ is at $\alpha_l=0.177$. But for the biased case $\epsilon\neq 0$ the system is always in the delocalized phase with finite $\eta>0$ (see Fig. 1 for $\epsilon=0.02$ and 0.05). We note that this is one of the most important differences between the biased and unbiased cases.

In the following, the transformed Hamiltonian is approximated as $\tilde{H}\approx\tilde{H}_0+\tilde{H}_1$ since $\langle g_0|\tilde{H}_2|g_0\rangle=0$ [because of the definition for η [Eq. (7)] and the terms in $\tilde{H}_2$ are related to the multiboson nondiagonal transition (like $b_k b_{k'}$ and $b_k^\dagger b_{k'}^\dagger$)]. The contributions of these nondiagonal terms to the physical quantities are $O(g_k^4)$. For zero temperature case the contribution from these multiboson nondiagonal transitions may be dropped safely.

III. DENSITY OPERATOR AND MASTER EQUATION

The density operator in Schrödinger representation is $\rho_{SB}(t)$ with Hamiltonian H, where the subscript SB indicates that it is density operator for the coupled two-level system and bath. For transformed Hamiltonian $\tilde{H}$ the density operator is $\tilde{\rho}_{SB}(t)=U^\dagger e^S\rho_{SB}(t)e^{-S}U$. We treat $\tilde{H}_0$ as the unperturbed Hamiltonian and the density operator in the interaction representation is [24]

$$\tilde{\rho}_{SB}^I(t)=\exp(i\tilde{H}_0 t)\tilde{\rho}_{SB}(t)\exp(-i\tilde{H}_0 t). \tag{19}$$

The equation of motion for $\tilde{\rho}_{SB}^I(t)$ is

$$\frac{d}{dt}\tilde{\rho}_{SB}^I(t)=-i[\tilde{H}_1(t),\tilde{\rho}_{SB}^I(t)]. \tag{20}$$

$\tilde{H}_1(t)$ is the perturbation $\tilde{H}_1$ in the interaction representation,

$$\begin{aligned}\tilde{H}_1(t)=&\frac{1}{2}(1-\sigma_z)\sum_k Q_k(b_k^\dagger e^{i\omega_k t}+b_k e^{-i\omega_k t})\\&+\frac{1}{2}\sum_k V_k[b_k^\dagger(\sigma_x+i\sigma_y)e^{i(\omega_k-W)t}\\&+b_k(\sigma_x-i\sigma_y)e^{-i(\omega_k-W)t}].\end{aligned} \tag{21}$$

We assume

$$\tilde{\rho}_{SB}^I(t)=\tilde{\rho}_S^I(t)\rho_B, \tag{22}$$

where $\tilde{\rho}_S^I(t)=\mathrm{Tr}_B\,\tilde{\rho}_{SB}^I(t)$ is the reduced density operator. Then we get the master equation [24] for $\tilde{\rho}_S^I(t)$,

$$\frac{d}{dt}\tilde{\rho}_S^I(t)=-\int_0^t \mathrm{Tr}_B\{\tilde{H}_1(t),[\tilde{H}_1(t'),\tilde{\rho}_S^I(t')\rho_B]\}dt', \tag{23}$$

where we neglect all higher order (than g_k^2) terms.

Because the density operator is Hermitian, i.e., $[\tilde{\rho}_S^I(t)]^\dagger=\tilde{\rho}_S^I(t)$, we can consider only two terms $\tilde{\rho}_{22}^I(t)$ and $\tilde{\rho}_{21}^I(t)$. Starting from Eq. (23), for zero temperature we can reach the following equations for them:

$$\frac{d}{dt}\tilde{\rho}_{22}^I(t)=-\int_0^t dt'\sum_k V_k^2[e^{-i(\omega_k-W)(t-t')}+e^{i(\omega_k-W)(t-t')}]\tilde{\rho}_{22}^I(t'), \tag{24}$$

$$\frac{d}{dt}\tilde{\rho}_{21}^I(t)=-\int_0^t dt'\sum_k [Q_k^2 e^{-i\omega_k(t-t')}+V_k^2 e^{-i(\omega_k-W)(t-t')}]\tilde{\rho}_{21}^I(t'). \tag{25}$$

Here the higher order terms are dropped. From Eq. (19) we have

$$\begin{pmatrix}\tilde{\rho}_{11}(t) & \tilde{\rho}_{12}(t)\\ \tilde{\rho}_{21}(t) & \tilde{\rho}_{22}(t)\end{pmatrix}=\begin{pmatrix}\tilde{\rho}_{11}^I(t) & \tilde{\rho}_{12}^I(t)e^{iWt}\\ \tilde{\rho}_{21}^I(t)e^{-iWt} & \tilde{\rho}_{22}^I(t)\end{pmatrix}, \tag{26}$$

then

$$\frac{d}{dt}\tilde{\rho}_{22}(t)=-\int_0^t dt'\sum_{\mathbf{k}} V_{\mathbf{k}}^2[e^{i(\omega_{\mathbf{k}}-W)(t-t')}+e^{-i(\omega_{\mathbf{k}}-W)(t-t')}]\tilde{\rho}_{22}(t'), \tag{27}$$

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}_{21}(t)=&-iW\tilde{\rho}_{21}(t)-\int_0^t dt'\sum_k [Q_k^2 e^{-i(\omega_k+W)(t-t')}\\&+V_k^2 e^{-i\omega_k(t-t')}]\tilde{\rho}_{21}(t').\end{aligned} \tag{28}$$

These equations can be solved by means of the Laplace transformation,

$$\tilde{\rho}_{22}(p) = \frac{\tilde{\rho}_{22}(0)}{p + \sum_k V_k^2 \left[\frac{1}{p - i(\omega_k - W)} + \frac{1}{p + i(\omega_k - W)} \right]}, \tag{29}$$

$$\tilde{\rho}_{21}(p) = \frac{\tilde{\rho}_{21}(0)}{p + iW + \sum_k \left[\frac{Q_k^2}{p + i(\omega_k + W)} + \frac{V_k^2}{p + i\omega_k} \right]}. \tag{30}$$

Then, the inverse Laplace transformation is performed with the so-called Bromwich path $\int_B$,

$$\tilde{\rho}_{22}(t) = \frac{1}{2\pi i}\int_B dp \exp(pt)\tilde{\rho}_{22}(p) = \frac{\tilde{\rho}_{22}(0)}{2\pi}\int_{-\infty}^{\infty} \frac{i \exp(-i\omega t) d\omega}{\omega - [R(W+\omega) - R(W-\omega)] + i[\gamma(W+\omega) + \gamma(W-\omega)]} \tag{31}$$

and

$$\tilde{\rho}_{21}(t) = \frac{1}{2\pi i}\int_B dp \exp(pt)\tilde{\rho}_{21}(p) = \frac{\tilde{\rho}_{21}(0)}{2\pi}\int_{-\infty}^{\infty} \frac{i \exp(-i\omega t) d\omega}{\omega - W - \Sigma(\omega) + i\Gamma(\omega)}. \tag{32}$$

Here the integration on the Bromwich path has been changed to that on the real axis $-\infty < \omega < \infty$ by the transform $p = 0^+ - i\omega$, with 0^+ as a positive infinitesimal. $R(\omega)$ and $\gamma(\omega)$ are real and imaginary parts of $\Sigma_k V_k^2/(\omega - i0^+ - \omega_k)$,

$$R(\omega) = (\eta\Delta)^2 \sum_{\mathbf{k}} \frac{\lambda_k^2}{(\omega - \omega_{\mathbf{k}})} = (\eta\Delta)^2 \int_0^\infty d\omega' \frac{J(\omega')}{(\omega - \omega')(\omega' + W)^2}, \tag{33}$$

$$\gamma(\omega) = \pi(\eta\Delta)^2 \sum_{\mathbf{k}} \lambda_k^2 \delta(\omega - \omega_{\mathbf{k}}) = \frac{\pi J(\omega)(\eta\Delta)^2}{(\omega + W)^2}, \tag{34}$$

where $J(\omega)$ is the spectral density. The following abbreviation has been used:

$$\Gamma(\omega) = \gamma(\omega) + \frac{\epsilon^2}{\eta^2\Delta^2}\gamma(\omega - W), \tag{35}$$

$$\Sigma(\omega) = R(\omega) + \frac{\epsilon^2}{\eta^2\Delta^2}R(\omega - W). \tag{36}$$

The initial density operator, $\tilde{\rho}_{21}(0)$ and $\tilde{\rho}_{22}(0)$, of the coupled system and its surrounding at $t=0$ is $\rho_{SB}(0) = e^{-S}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)\rho_B e^S$, and the corresponding initial reduced density operator for $\tilde{H} \approx \tilde{H}_0 + \tilde{H}_1$ is

$$\tilde{\rho}_S(0) = \frac{1}{2}\begin{pmatrix} 1 - \epsilon/W & \eta\Delta/W \\ \eta\Delta/W & 1 + \epsilon/W \end{pmatrix}. \tag{37}$$

IV. NONEQUILIBRIUM CORRELATION

One of our purpose is to calculate the nonequilibrium correlation $P(t)$, which is defined as $P(t) = \mathrm{Tr}_S\{\mathrm{Tr}_B[\rho_{SB}(t)\sigma_z]\}$, where $\rho_{SB}(t)$ is the density operator for the original Hamiltonian. Because of the unitary transforms, it can be calculated as

$$P(t) = \mathrm{Tr}_S\{\mathrm{Tr}_B[e^{-S}U\tilde{\rho}_{SB}(t)U^\dagger e^S \sigma_z]\} = \mathrm{Tr}_S\left(\tilde{\rho}_S(t)\left[-\frac{\epsilon}{W}\sigma_z + \frac{\eta\Delta}{W}\sigma_x\right]\right) = \frac{\epsilon}{W}[2\tilde{\rho}_{22}(t) - 1] + \frac{2\eta\Delta}{W}\mathrm{Re}[\tilde{\rho}_{21}(t)]. \tag{38}$$

Thus, Eqs. (31) and (32) are to be calculated to get $P(t)$ by numerical integration with sub-Ohmic spectral density.

The Markovian approximation is equivalent to approximate the integration by the residue theorem with the simple pole of the integrand of Eq. (31) at $-2i\gamma_0$ and that of Eq. (32) at $\omega_0 - i\gamma_0$, which leads to

$$P(t) = \frac{\epsilon}{W}\left\{\left(\frac{\epsilon}{W} + 1\right)\exp[-2\gamma_0 t] - 1\right\} + \frac{\eta^2\Delta^2}{W^2}\cos(\omega_0 t)\exp[-\gamma_0 t)], \tag{39}$$

where $\gamma_0 = \gamma(W)$ is the Weisskopf-Wigner approximation for the decay rate,

$$\gamma(W) = \frac{1}{2}\pi\alpha\omega_c^{1-s}W^s\frac{(\eta\Delta)^2}{W^2}, \tag{40}$$

and $\omega_0 = W + \Sigma(W)$ [$\Sigma(W)$ is the level shift]. Figure 2(a) shows comparison between Eqs. (38) and (39) for weak-coupling case $\alpha = 0.02$ ($s = 1/2$, $\Delta/\omega_c = 0.1$, and $\epsilon/\omega_c = 0.05$), where our non-Markovian dynamics [Eq. (38)] is nearly the same as that of the Markovian approximation [Eq. (39)]. However, for strong-coupling case $\alpha = 0.1$, Fig. 2(b) shows a big difference of our non-Markovian dynamics from that of the Markovian one, especially for the short time oscillating

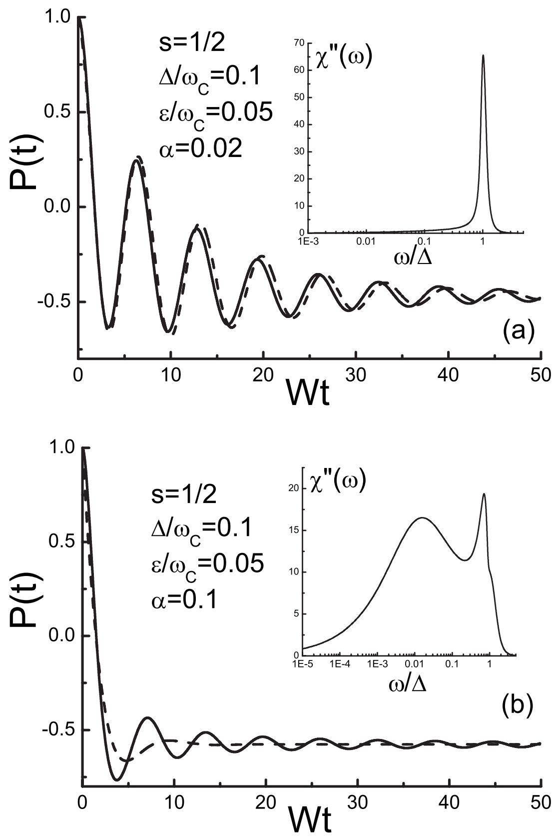

FIG. 2. Nonequilibrium correlation $P(t)$ for (a) weak-coupling case $\alpha=0.02$ and (b) strong-coupling case $\alpha=0.1$. The dashed lines are the Markovian approximation [Eq. (39)]. The insets show the susceptibility $\chi''(\omega)$ vs ω relations.

behavior which has been smeared out in Markovian approximation. The insets are for the susceptibility $\chi''(\omega)$ which will be calculated in Sec. V. Here we note that for weak-coupling case [Fig. 2(a)] the susceptibility has one sharp peak only and the Markovian approximation may be a good one; for strong-coupling case [Fig. 2(b)] the susceptibility has two peaks and the Markovian approximation is not good at least for the short time behavior.

A nonzero bias plays an important role in the quantum dynamics of the two-level system coupled with a sub-Ohmic bath. Figure 3 shows the nonequilibrium correlation $P(t)$ for zero and nonzero biases. One can see that, apart from the effect of bias on the behavior of long time limit $[P(t\rightarrow\infty)=-\epsilon/W]$, a nonzero bias enhances the quantum coherence as the decay rate of the Rabi oscillation for the case of $\epsilon/\omega_c=0.05$ is obviously lower than that of $\epsilon=0$.

Figure 4 compares the nonequilibrium correlation $P(t)$ for different couplings for $s=1/2$ with nonzero bias $\epsilon/\omega_c=0.02$. For weak coupling $\alpha=0.01$ (dashed-dotted line), the quantum coherence may be kept for a longer time, but for the moderate coupling $\alpha=0.04$ (solid line), the Rabi oscillation proceeds for a shorter time. For the strong coupling $\alpha=0.091$ (dashed line), the Rabi oscillation is quite weak. We note that for $s=1/2$ and $\epsilon/\omega_c=0.02$ the coherence-decoherence transition point is at $\alpha_c=0.0917$, which will be determined in Sec. V.

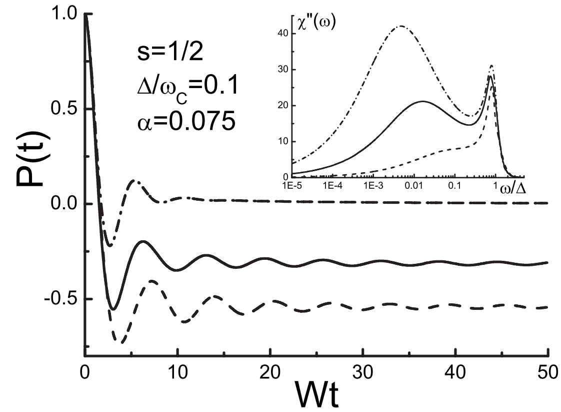

FIG. 3. Nonequilibrium correlation $P(t)$ for different biases: $\epsilon/\omega_c=0$ (dashed-dotted line), 0.025 (solid line), and 0.05 (dashed line). The inset shows the susceptibility $\chi''(\omega)$ vs ω relations for different biases.

In our approach two approximations have been made: one is the omission of $\widetilde{H}_2$ and the other is the usual Born approximation for deriving the master equation [Eq. (23)]. Hence, the validity of our approach should be checked and one check may be the sum rule $P(t=0)=1$ for Eq. (38). It has been checked and is satisfied exactly for all the cases we calculated.

V. SUSCEPTIBILITY AND COHERENCE-DECOHERENCE TRANSITION

The retarded Green's functions [25] are

$$G(t) = -i\theta(t)Z^{-1}\,\mathrm{Tr}\{\exp(-\beta H)[\exp(iHt)\sigma_z\exp(-iHt),\sigma_z]\}, \tag{41}$$

where $[A,B]=AB-BA$ and $Z=\mathrm{Tr}[\exp(-\beta H)]$. The Fourier transformation $G(\omega)$ is obtained in the Appendix. The imaginary part of $G(\omega)$ is

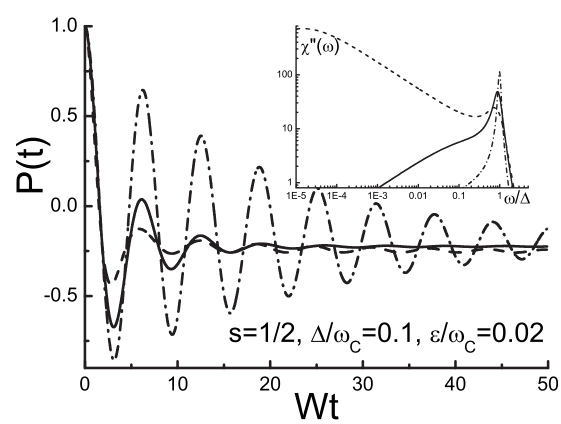

FIG. 4. Nonequilibrium correlation $P(t)$ for different couplings: alpha=0.01 (dashed-dotted line), 0.04 (solid line), and 0.091 (dashed line). The inset shows the susceptibility $\chi''(\omega)$ vs ω relations for different couplings. The coherence-decoherence transition point is at $\alpha_c=0.0917$.

TABLE I. Shiba's relation is checked for nonzero bias and sub-Ohmic cases. $R=\lim_{\omega\to 0} S(\omega)/\frac{\pi}{4}[\chi'(0)]^2$, where $S(\omega)=\chi''(\omega)/J(\omega)$.

s	Δ/ω_C	ϵ/ω_C	α	$\chi'(0)$	$\lim_{\omega\to 0} S(\omega)$	R
1/4	0.1	0	0.01	34.495979	934.60225	1.0
1/4	0.1	0.05	0.03	121.46668	11587.889	1.0000004
1/4	0.1	0.1	0.05	23.147908	420.84016	1.0000088
1/2	0.1	0	0.05	51.586893	2090.1075	1.0
1/2	0.1	0.05	0.1	73.517115	4244.8999	1.0000015
1/2	0.1	0.1	0.2	19.755668	306.53734	1.0000232
3/4	0.1	0	0.1	43.000554	1452.2386	1.0
3/4	0.1	0.05	0.3	78.204433	4803.4538	1.0000023
3/4	0.1	0.1	0.5	11.44906	102.95554	1.0000465
1	0.1	0	0.5	3072.9597	7416578.9	1.0
1	0.1	0.01	0.5	694.5864	378915.56	1.0
1	0.1	0.02	0.5	205.86769	33286.362	1.0000002

$$\mathrm{Im}\, G(\omega) = -\frac{(\eta\Delta)^2}{W^2}\langle\sigma_z\rangle_{\tilde{H}_0}\left\{\frac{\Gamma(\omega)\theta(\omega)}{[\omega - W - \Sigma(\omega)]^2+\Gamma^2(\omega)} - \frac{\Gamma(-\omega)\theta(-\omega)}{[\omega + W + \Sigma(-\omega)]^2+\Gamma^2(-\omega)}\right\}. \quad (42)$$

$\langle\sigma_z\rangle_{\tilde{H}_0}=1$, which is the average value of σ_z in the ground state of $\tilde{H}_0$. $\Sigma(\omega)$ and $\Gamma(\omega)$ are those in Eqs. (35) and (36).

The susceptibility $\chi(\omega)=-G_{-}(\omega)$ and its imaginary part is

$$\chi''(\omega) = \int_{-\infty}^{\infty} dt \exp(i\omega t)\frac{1}{2}\mathrm{Tr}\{\exp(-\beta H)[\sigma_z(t)\sigma_z - \sigma_z\sigma_z(t)]\}/Z$$
$$= \frac{(\eta\Delta)^2}{W^2}\left\{\frac{\Gamma(\omega)\theta(\omega)}{[\omega - W - \Sigma(\omega)]^2+\Gamma^2(\omega)} - \frac{\Gamma(-\omega)\theta(-\omega)}{[\omega + W + \Sigma(-\omega)]^2+\Gamma^2(-\omega)}\right\}. \quad (43)$$

The real part of static susceptibility $\chi'(\omega=0)$ can be obtained by the following integral:

$$\chi'(\omega=0) = \frac{2}{\pi}\int_0^{\infty}\frac{\chi''(\omega)}{\omega}d\omega. \quad (44)$$

Another check for our approach is Shiba's relation [3,6,7],

$$\lim_{\omega\to 0}\frac{\chi''(\omega)}{J(\omega)} = \frac{\pi}{4}[\chi'(\omega=0)]^2, \quad (45)$$

which should be satisfied for the two-level system coupled to a heat bath. Table I shows that Shiba's relation is exactly satisfied for $s\leq 1$ in our calculations.

Generally speaking, $\chi''(\omega)$ is an even function of ω, $\chi''(-\omega)=\chi''(\omega)$. For the Ohmic bath ($s=1$) with zero bias, it is well known [1,2] that $\chi''(\omega)$ has a peak at $\omega=\Delta_r$ for $\alpha<\alpha_c=1/2$, where Δ_r is the renormalized tunneling. Besides, $\chi''(\omega=0)=0$ when $\alpha<\alpha_c=1/2$. At $\alpha=\alpha_c$, the peak moves to $\omega=0$ with $\chi''(\omega=0)=\infty$. α_c is the coherence-decoherence transition point [1,2].

For the Ohmic bath ($s=1$) with zero bias ($\epsilon=0$) our result is exactly the same, that is, $\chi''(\omega)$ has a peak at $\omega\approx\omega_p$ for $\alpha<\alpha_c$ with $\alpha_c=\frac{1}{2}(1+\eta\Delta/\omega_c)$ [13,23], where ω_p is the solution of following equation:

$$\omega_p - W - \Sigma(\omega_p) = 0. \quad (46)$$

But for nonzero bias and sub-Ohmic bath, which is the main focus of this work, the spectrum of $\chi''(\omega)$ becomes more complicated. When coupling is weak (smaller α), $\chi''(\omega)$ has one peak only at $\omega\approx\omega_p$ [see the inset of Fig. 2(a)]. For stronger coupling, $\chi''(\omega)$ has two peaks in the range $\omega>0$, one is at $\omega\approx\omega_p$ [the solution of Eq. (46)] and the other at the place much lower than ω_p [see the inset of Fig. 2(b)].

The inset of Fig. 3 shows the effect of a nonzero bias on the susceptibility $\chi''(\omega)$. When ϵ/ω_c increases, the double peak structure of the susceptibility $\chi''(\omega)$ for $\epsilon=0$ changes gradually to a single peak structure for $\epsilon/\omega_c=0.05$. Besides, the inset of Fig. 4 shows the effect of different couplings on the susceptibility $\chi''(\omega)$ for $s=1/2$ and nonzero bias $\epsilon/\omega_c=0.02$. For weak coupling $\alpha=0.01$ (dashed-dotted line), the quantum coherence can be kept for a longer time and the susceptibility has a sharp single peak. But for the moderate coupling $\alpha=0.04$ (solid line), the second peak with lower frequency emerges. For the strong coupling $\alpha=0.091$ (dashed line), the second peak with lower frequency approaches $\omega\approx 0$ with higher peak height.

At $\alpha=\alpha_c$, the second peak is at $\omega=0$ with infinite height $\chi''(\omega=0)=\infty$ since $\Gamma(\omega)\sim\omega^s$ for $\omega\to 0$. That means, at $\alpha=\alpha_c$, the lower peak moves to $\omega=0$ while the higher peak may still exist. Here, as in the case of zero bias and Ohmic bath, we define α_c as the coherence-decoherence transition point for nonzero bias and sub-Ohmic bath. Mathematically, α_c is determined as the solution of

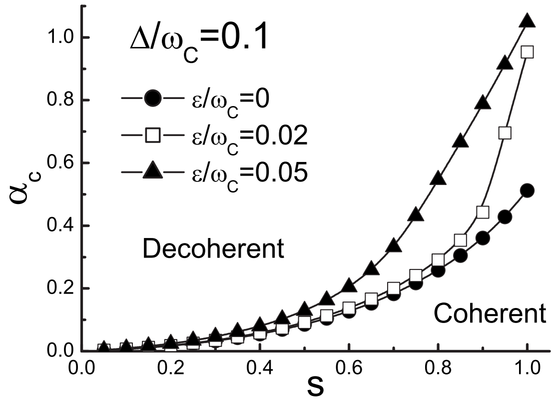

FIG. 5. The coherent-decoherent transition point α_c vs the index of sub-Ohmic bath s relations for different biases: $\epsilon/\omega_c=0$ (solid circles), 0.02 (empty squares), and 0.05 (solid triangles).

$$-W-\Sigma(0)=0 \tag{47}$$

because this is the condition for $\chi''(\omega=0)=\infty$. For $s<1$ the formula for α_c is quite complicated, but for nonzero bias with Ohmic bath ($s=1$) it is

$$\alpha_c=\frac{1}{2}\left[\frac{(\eta\Delta)^2}{W^2}\frac{\omega_c}{\omega_c+W}+\frac{1}{2}\frac{\epsilon^2}{W^2}\frac{\omega_c^2}{(\omega_c+W)^2}\right]^{-1}, \tag{48}$$

which leads to the transition point for zero bias ($\epsilon=0$): $\alpha_c=\frac{1}{2}(1+\eta\Delta/\omega_c)$ [23].

Figure 5 shows the coherence-decoherence transition point α_c as functions of the index of sub-Ohmic bath s, which may be treated as a "phase diagram" with the area of $\alpha<\alpha_c$ as the "coherent phase" but that of $\alpha>\alpha_c$ the "decoherent phase."

VI. CONCLUDING REMARKS

The non-Markovian dynamics of the dissipative two-level system coupled to the sub-Ohmic bath ($0<s\leq 1$) with nonzero bias has been studied by means of the perturbation approach based on a unitary transformation. It has been shown that the Markovian approximation may be not good for the case of sub-Ohmic bath and the non-Markovian approach should be used especially for the short time behavior of the coupled system and environment. The nonequilibrium correlation $P(t)$ has been calculated to show that a finite bias may favor the short time coherence. The spectrum of the susceptibility $\chi''(\omega)$ of the sub-Ohmic case may have a double peak structure in the range of $\omega>0$ when the coupling α is relatively strong. Besides, the coherence-decoherence transition point α_c is determined for different $0<s\leq 1$ by the condition of $\chi''(\omega=0)=\infty$ when $\alpha=\alpha_c$. Our results have been checked by showing that Shiba's relation is exactly satisfied for $0<s\leq 1$ with nonzero bias.

The key point of our treatment is the unitary transformation with generator equation [Eq. (2)], where a parameter ξ_k is introduced. After the transformation a perturbation expansion has been performed. If $\xi_k=0$ for all k, that is, without the transformation, the perturbation expansion would be similar to the standard weak-coupling expansion (Bloch-Redfield theory). Besides, if $\xi_k=1$ for all k, then our transformation is the usual polaronic transformation and the perturbation expansion is for the small parameter Δ which is equivalent to the non-interacting blip approximation [26]. Our choice for $0<\xi_k<1$ [Eq. (15)] is in between and thus is an improvement on the analytical methods.

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APPENDIX

The retarded Green's functions are

$$\begin{aligned}G(t)&=-i\theta(t)Z^{-1}\,\mathrm{Tr}\{\exp(-\beta H)[\exp(iHt)\sigma_z\exp(-iHt),\sigma_z]\}\\&=-i\theta(t)Z^{-1}\,\mathrm{Tr}\{\exp(-\beta H')[\exp(iH't)\sigma_z\exp(-iH't),\sigma_z]\}\\&=-i\theta(t)Z^{-1}\,\mathrm{Tr}\{\exp(-\beta\tilde{H})[\exp(i\tilde{H}t)U^{\dagger}\sigma_zU\exp(-i\tilde{H}t),U^{\dagger}\sigma_zU]\}\\&=-i\theta(t)Z^{-1}\,\mathrm{Tr}\left\{\exp(-\beta\tilde{H})\left(\frac{\epsilon^2}{W^2}[\sigma_z(t),\sigma_z]+\frac{(\eta\Delta)^2}{W^2}[\sigma_x(t),\sigma_x]-\frac{\epsilon\eta\Delta}{W^2}[\sigma_z(t),\sigma_x]-\frac{\epsilon\eta\Delta}{W^2}[\sigma_x(t),\sigma_z]\right)\right\},\end{aligned} \tag{A1}$$

where

$$Z=\mathrm{Tr}[\exp(-\beta H)]=\mathrm{Tr}[\exp(-\beta\tilde{H})],$$

$[A,B]=AB-BA$, and $\sigma_{z(x)}(t)=\exp(i\tilde{H}t)\sigma_{z(x)}\exp(-i\tilde{H}t)$ is in the Heisenberg picture of $\tilde{H}$. The Fourier transformation of $G(t)$ is denoted as $G(\omega)$,

$$G(\omega)=\frac{\epsilon^2}{W^2}\langle\langle\sigma_z;\sigma_z\rangle\rangle+\frac{(\eta\Delta)^2}{W^2}\langle\langle\sigma_x;\sigma_x\rangle\rangle-\frac{\epsilon\eta\Delta}{W^2}\langle\langle\sigma_z;\sigma_x\rangle\rangle-\frac{\epsilon\eta\Delta}{W^2}\langle\langle\sigma_x;\sigma_z\rangle\rangle,$$

where

$$\langle\langle A;B\rangle\rangle = -i\theta(t)Z^{-1}\,\mathrm{Tr}\{\exp(-\beta\widetilde{H})[\exp(i\widetilde{H}t)A\,\exp(-i\widetilde{H}t),B]\}$$

denotes the retarded Green's function which satisfies the following equation of motion:

$$\omega\langle\langle A;B\rangle\rangle = \langle[A,B]\rangle + \langle\langle[A,\widetilde{H}];B\rangle\rangle,$$

$$\langle[A,B]\rangle = Z^{-1}\,\mathrm{Tr}\{\exp(-\beta\widetilde{H})[A,B]\}.$$

Thus, we can get the following equation chain:

$$\omega\langle\langle\sigma_x;\sigma_x\rangle\rangle = W\langle\langle i\sigma_y;\sigma_x\rangle\rangle + \sum_k Q_k\langle\langle i\sigma_y(b_k^\dagger + b_k);\sigma_x\rangle\rangle - \sum_k V_k\langle\langle\sigma_z(b_k^\dagger - b_k);\sigma_x\rangle\rangle, \quad (A2)$$

$$\omega\langle\langle i\sigma_y;\sigma_x\rangle\rangle = 2\langle\sigma_z\rangle + W\langle\langle\sigma_x;\sigma_x\rangle\rangle + \sum_k Q_k\langle\langle\sigma_x(b_k^\dagger + b_k);\sigma_x\rangle\rangle + \sum_k V_k\langle\langle\sigma_z(b_k^\dagger + b_k);\sigma_x\rangle\rangle, \quad (A3)$$

$$\omega\langle\langle\sigma_x(b_k^\dagger + b_k);\sigma_x\rangle\rangle = -\omega_k\langle\langle\sigma_x(b_k^\dagger - b_k);\sigma_x\rangle\rangle + W\langle\langle i\sigma_y(b_k^\dagger + b_k);\sigma_x\rangle\rangle + Q_k\langle\langle i\sigma_y;\sigma_x\rangle\rangle, \quad (A4)$$

$$\omega\langle\langle\sigma_x(b_k^\dagger - b_k);\sigma_x\rangle\rangle = -\omega_k\langle\langle\sigma_x(b_k^\dagger + b_k);\sigma_x\rangle\rangle + W\langle\langle i\sigma_y(b_k^\dagger - b_k);\sigma_x\rangle\rangle - Q_k\langle\langle\sigma_x;\sigma_x\rangle\rangle, \quad (A5)$$

$$\omega\langle\langle i\sigma_y(b_k^\dagger + b_k);\sigma_x\rangle\rangle = -\omega_k\langle\langle i\sigma_y(b_k^\dagger - b_k);\sigma_x\rangle\rangle + W\langle\langle\sigma_x(b_k^\dagger + b_k);\sigma_x\rangle\rangle + Q_k\langle\langle\sigma_x;\sigma_x\rangle\rangle, \quad (A6)$$

$$\omega\langle\langle i\sigma_y(b_k^\dagger - b_k);\sigma_x\rangle\rangle = -\omega_k\langle\langle i\sigma_y(b_k^\dagger + b_k);\sigma_x\rangle\rangle + W\langle\langle\sigma_x(b_k^\dagger - b_k);\sigma_x\rangle\rangle - Q_k\langle\langle i\sigma_y;\sigma_x\rangle\rangle, \quad (A7)$$

where $V_k = \eta\Delta g_k\xi_k/\omega_k$ and $Q_k = \epsilon g_k\xi_k/\omega_k$. We already made the cutoff approximation for the equation chain at the second order of g_k. Besides, until the second order of g_k we have $\langle\langle\sigma_z;\sigma_x\rangle\rangle = 0$, $\langle\langle\sigma_z;\sigma_z\rangle\rangle = 0$, and $\langle\langle\sigma_x;\sigma_z\rangle\rangle = 0$. So the solution for $G(\omega)$ is

$$G(\omega) = \frac{(\eta\Delta)^2}{W^2}\times\left(\frac{\langle\sigma_z\rangle}{\omega - W - \sum_k V_k^2/(\omega-\omega_k) - \sum_k Q_k^2/(\omega - W - \omega_k)} - \frac{\langle\sigma_z\rangle}{\omega + W - \sum_k V_k^2/(\omega+\omega_k) - \sum_k Q_k^2/(\omega + W + \omega_k)}\right). \quad (A8)$$

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